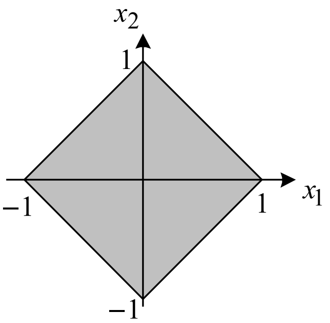
Metric and Topological Spaces



1. (i) (a) $f^{-1}([-1,\ 2])$

$$=\left\{(x_1,\ x_2)\in\mathbb{R}^2 : f(x_1,\ x_2)\in[-1,\ 2]\right\}$$
$$=\left\{(x_1,\ x_2)\in\mathbb{R}^2 : -1\le 1+|x_1|+|x_2|\le 2\right\}$$
$$=\left\{(x_1,\ x_2)\in\mathbb{R}^2 : |x_1|+|x_2|\le 1\right\}.$$

(b) $$f\left(f^{-1}([-1,2])\right)=f\left(\left\{(x_1,x_2)\in\mathbb{R}^2 : |x_1|+|x_2|\le 1\right\}\right)$$
$$=\left\{f(x_1,x_2) : |x_1|+|x_2|\le 1\right\}$$
$$=\left\{1+|x_1|+|x_2| : |x_1|+|x_2|\le 1\right\}$$
$$=[1,2].$$

(ii) (a) Let $y\in g(g^{-1}(K))$. Then $y=g(x)$ for some $x\in g^{-1}(K)$. But $x\in g^{-1}(K)$ implies that $g(x)\in K$; that is, $y\in K$.

We have proved: $y\in g(g^{-1}(K)) \Rightarrow y\in K$; therefore $g(g^{-1}(K))\subseteq K$.

(b) Suppose that g is onto and let $y\in K$. Since g is onto, $y=g(x)$ for some $x\in A$. Since $g(x)\in K$, we have $x\in g^{-1}(K)$ so that $g(x)\in g(g^{-1}(K))$; that is, $y\in g(g^{-1}(K))$.

We have proved: $y\in K \Rightarrow y\in g(g^{-1}(K))$; therefore $K\subseteq g(g^{-1}(K))$.

From part (a) we can conclude $K=g(g^{-1}(K))$.

2. (i) If $x=y$ then $d(x,y)=0$, by definition. If $x\ne y$ then $d(x,y)\ge |x-y|>0$. Therefore d satisfies (M1).

If $x\ne y$ then
$$d(x,y)=|x+1|+|y+1|+|x-y|=|y+1|+|x+1|+|y-x|=d(y,x).$$

Therefore d satisfies (M2).

Let x, y, z be distinct elements of $\mathbb{R}$. Then

$$\begin{aligned} d(x,\ y)+d(y,\ z) &= |x+1|+|y+1|+|x-y|+|y+1|+|z+1|+|y-z| \\ &\ge |x+1|+|z+1|+|x-y|+|y-z| && \text{(since } |y+1|\ge 0\text{)} \\ &\ge |x+1|+|z+1|+|x-z| && \text{(triangle inequality)} \\ &= d(x,\ z). \end{aligned}$$

Therefore d satisfies (M3). Hence d is a metric on $\mathbb{R}$.

(ii) (a) $B_1(0) = \{x \in \mathbb{R} : d(x,\ 0) < 1\}$

$$= \{0\} \cup \left\{x \in \mathbb{R} : |x+1| + 1 + |x| < 1\right\}$$
$$= \{0\} \cup \varnothing$$
$$= \{0\}.$$

(b) $B_1(-1) = \left\{x \in \mathbb{R} : d(x,\ -1) < 1\right\}$

$$= \{-1\} \cup \left\{x \in \mathbb{R} : |x+1| + |-1+1| + |x-(-1)| < 1\right\}$$
$$= \{-1\} \cup \left\{x \in \mathbb{R} : |x+1| < \tfrac{1}{2}\right\}$$
$$= \left(-\tfrac{3}{2}, -\tfrac{1}{2}\right).$$

(iii) By a similar argument to part (ii) (b), we see that, for any $r > 0$,

$$B_r(-1) = \left(-1 - \frac{r}{2}, -1 + \frac{r}{2}\right).$$

Therefore, for all $r > 0$,

$$f\big(B_r(-1)\big) = \left(-\frac{r}{2}, \frac{r}{2}\right) \not\subset \{0\} = B_1(0) = B_1(f(-1)).$$

Hence f is not (d, d)-continuous at $x = -1$.

3. (i) $\varnothing \in \mathscr{T}$ since $0 \notin \varnothing$ and $A \in \mathscr{T}$ since $B \subset A$. Therefore $\mathscr{T}$ satisfies (T1).

Let $U, V \in \mathscr{T}$.

If $0 \notin U$ or $0 \notin V$ then $0 \notin U \cap V$ so $U \cap V \in \mathscr{T}$.

Otherwise $B \subset U$ and $B \subset V$ so $B \subset U \cap V$ so $U \cap V \in \mathscr{T}$.

Therefore $\mathscr{T}$ satisfies (T2).

Let $\{V_i\}_{i \in I}$ be a family of sets in $\mathscr{T}$.

If $0 \notin U_i$ for all $i \in I$, then $0 \notin \bigcup_{i \in I} U_i$ so $\bigcup_{i \in I} U_i \in \mathscr{T}$.

Otherwise $B \subset U_j$ for some $j \in I$. Then $B \subset \bigcup_{i \in I} U_i$ so $\bigcup_{i \in I} U_i \in \mathscr{T}$.

Therefore $\mathscr{T}$ satisfies (T3).

Hence $\mathscr{T}$ is a topology on A.

(ii) Let $X = \{0, 1, 2, 3\}$ and let $\mathscr{T}_X$ denote the induced topology on X.

If $0 \notin U$ then $U \in \mathscr{T}$ so $U = U \cap X \in \mathscr{T}_X$.

If $0 \in U$ and $U \in \mathscr{T}_X$ then $U = X \cap V$ where $B \subset V$. Therefore $1 \in U$ and $3 \in U$.

Therefore $U \in \mathscr{T}_X$ if and only if $0 \notin U$ or $\{0, 1, 3\} \subset U$.

Hence $\mathscr{T}_X = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}, \{0, 1, 3\}, X\}$.

4. Note that, for $U \in \mathscr{V}$, $0 \in U \Rightarrow 1 \in U$.

   (i) Let $U \in \mathscr{V}$. There are three cases to consider.

   Case 1: $0 \notin U$, $1 \notin U$; then $f^{-1}(U) = \varnothing \in \mathscr{T}$.

   Case 2: $0 \notin U$, $1 \in U$; then $f^{-1}(U) = \{a, b\} \in \mathscr{T}$.

   Case 3: $0 \in U$, $1 \in U$; then $f^{-1}(U) = \{a, b, c\} = A \in \mathscr{T}$.

   Therefore $U \in \mathscr{V} \Rightarrow f^{-1}(U) \in \mathscr{T}$ so f is $(\mathscr{T}, \mathscr{V})$-continuous.

   (ii) $\{1\} \in \mathscr{V}$ but $g^{-1}(\{1\}) = \{b, c\} \notin \mathscr{T}$. Therefore g is **not** $(\mathscr{T}, \mathscr{V})$-continuous.

   (iii) $\{0, 1\} \in \mathscr{V}$ but $h^{-1}(\{0,1\}) = \{b, c\} \notin \mathscr{T}$. Therefore h is **not** $(\mathscr{T}, \mathscr{V})$-continuous.

5. (i) Let $x \in A$, $x > \frac{1}{3}$ and let U be an open set containing x.

   Then $U = (a, b)$ for some $a < 0$, $x < b < 1$. Therefore $\frac{1}{3} \in U$ so U contains a point of $H - \{x\}$.
   Therefore x is a limit point of H.

   (ii) Let $x = \frac{1}{3}$. Then $\left(-1, \frac{1}{2}\right)$ is an open set containing x which contains no point of $H - \{x\} = \left\{\frac{1}{2}\right\}$. Therefore x is not a limit point of H.

   Let $x \in A$, $x < \frac{1}{3}$. Then $\left(-1, \frac{1}{3}\right)$ is an open set containing x which contains no point of $H - \{x\} = H$. Therefore x is not a limit point of H.
   Hence if $x \in A$, $x \leq \frac{1}{3}$ then is not a limit point of H.

   (iii)
   $$\begin{aligned} \mathrm{Cl}(H) &= H \cup \{\text{limit points of } H\} \\ &= \left\{\tfrac{1}{3}, \tfrac{1}{2}\right\} \cup \left(\tfrac{1}{3}, 1\right) \\ &= \left[\tfrac{1}{3}, 1\right) \end{aligned}$$

6. (i) Let X be a compact space and let A be a closed subspace of X.
   Let $\mathscr{U}$ be an open cover of A; that is, $\mathscr{U} = \{U_i\}_{i \in I}$ is a collection of sets, open in X, such that $A \subset \bigcup_{i \in I} U_i$.

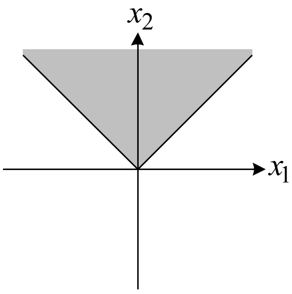
   Then $\mathscr{U} \cup \{X - A\}$ is an open cover of X (since A is closed in X).

   Since X is compact, this cover has a finite subcover $\{U_1, U_2, \ldots, U_n, X - A\}$.

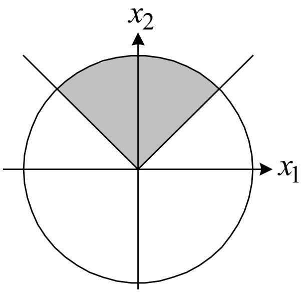
   Therefore $A \subset U_1 \cup U_2 \cup \ldots \cup U_n$ so $\{U_1, U_2, \ldots, U_n\}$ is a finite subcover of the cover $\mathscr{U}$ of A.

   Therefore A is compact.

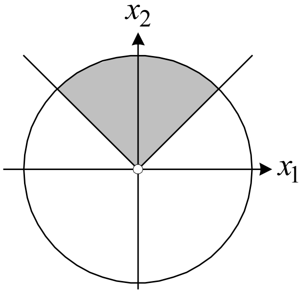
(ii)



H_1



H_2



H_3

H_1 is not bounded and therefore not compact.

H_2 is closed and bounded and is therefore compact.

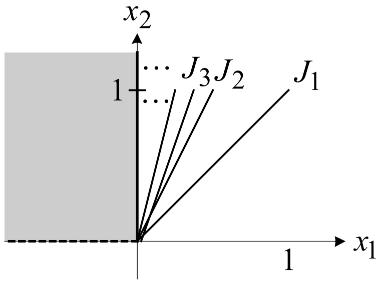
H_3 is not closed and therefore not compact.

7. (i) Let $f : A_1 \cup A_2 \to \{0, 1\}$ be a continuous function where $\{0, 1\}$ has the discrete topology. Then $f \mid A_1$ and $f \mid A_2$ are continuous. Since A_1 and A_2 are connected each of these restrictions $f \mid A_1$ and $f \mid A_2$ are constant. In other words, $f(A_1) = \{\alpha\}$ and $f(A_2) = \{\beta\}$ where $\alpha, \beta \in \{0, 1\}$.

But $A_1 \cap A_2 \neq \varnothing$ so $\alpha = \beta$. Hence $f(A_1 \cup A_2) = \{\alpha\}$ so f is constant.

Therefore $A_1 \cup A_2$ is connected.

(ii)



Each J_n, $n \in \mathbb{N}$, is homeomorphic to an interval and hence is connected. For all $n \neq m$, $J_n \cap J_m \neq \varnothing$; therefore $J = \bigcup_{n=1}^{\infty} J_n$ is connected by the generalization of part (i) to arbitrary collections.

K is connected and $(0, 0)$ is a limit point of K (every open set containing $(0, 0)$ also contains points of K). Therefore $K \subset K \cup \{(0, 0)\} \subset \text{Cl}(K)$ so $K \cup \{(0, 0)\}$ is connected by Proposition 6.2.18 (handbook page 10).

Now J and $K \cup \{(0, 0)\}$ are connected and $J \cap (K \cup \{(0, 0)\}) \neq \varnothing$. Therefore, by part (i), $\left(\bigcup_{n=1}^{\infty} J_n\right) \cup K = J \cup \left(K \cup \{(0, 0)\}\right)$ is connected.

8. (i) Suppose $m > n$. Then

$$\begin{aligned} d(x_n,\ x_m) &= d(n,\ m) \\ &= \left|\frac{n}{n+1} - \frac{m}{m+1}\right| \\ &= \left|\left(1-\frac{1}{n+1}\right)-\left(1-\frac{1}{m+1}\right)\right| \\ &= \left|\frac{1}{m+1} - \frac{1}{n+1}\right| \\ &= \frac{1}{n+1} - \frac{1}{m+1} \\ &< \frac{1}{n+1}. \end{aligned}$$

Let $\varepsilon > 0$. Choose N such that $\dfrac{1}{N+1} < \varepsilon$. (In other words, choose $N > \dfrac{1}{\varepsilon} - 1$.)

Then, for all $m > n > N$, $d(x_n, x_m) < \dfrac{1}{n+1} < \dfrac{1}{N+1} < \varepsilon$. Therefore (x_n) is a Cauchy sequence.

(ii) Suppose $n \geq 2k + 1$. Then

$$\begin{aligned} d\left(k, x_n\right) &= \left|\frac{k}{k+1} - \frac{n}{n+1}\right| \\ &= \left|\frac{1}{n+1} - \frac{1}{k+1}\right| && \text{(as above)} \\ &= \frac{1}{k+1} - \frac{1}{n+1} \\ &\geq \frac{1}{k+1} - \frac{1}{2k+2} && \left(\text{since } n \geq 2k+1 \Rightarrow \frac{1}{n+1} \leq \frac{1}{(2k+1)+1} = \frac{1}{2k+2}\right) \\ &= \frac{1}{2k+2}. \end{aligned}$$

(iii) By part (ii), the sequence (x_n) does not converge to k for any positive integer k.

[*Proof*: Choose $\varepsilon > 0$ such that $\varepsilon < \dfrac{1}{2k+2}$. Then for all sufficiently large n (i.e. $n \geq 2k + 1$), we have $d(x_n, k) \geq \dfrac{1}{2k+2} > \varepsilon$.]

Therefore the sequence (x_n) does not converge.
By part (i), (x_n) is a Cauchy sequence.

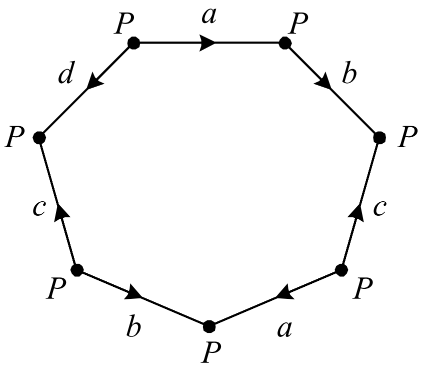
In a complete metric space, every Cauchy sequence converges. Therefore $\{\mathbb{N}, d\}$ is not a complete metric space.

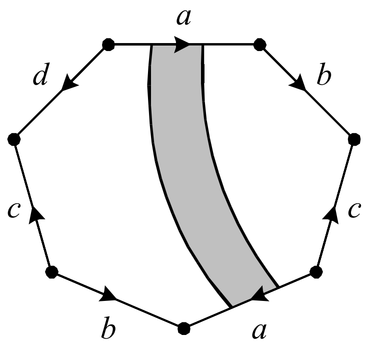
Geometric Topology

9. (i) $V = 1$, $E = 4$, $F = 1$; hence $\chi = -2$.

There is only one boundary edge so $\beta = 1$.

Hence the surface is $3\mathbb{R}P^2 \# D^2$.

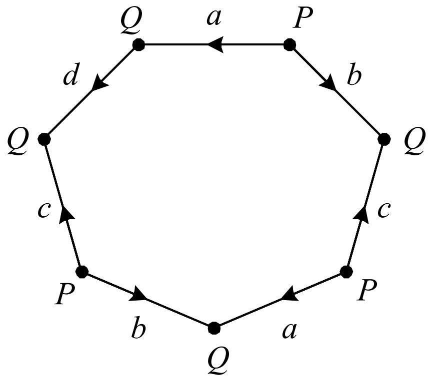




(ii) $V = 2$, $E = 4$, $F = 1$; hence $\chi = -1$.

There is only one boundary edge so $\beta = 1$.

Hence the surface is $T^2 \# D^2$.

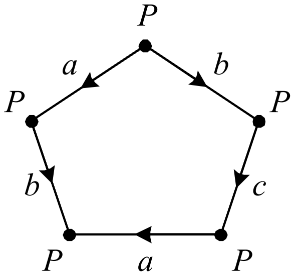


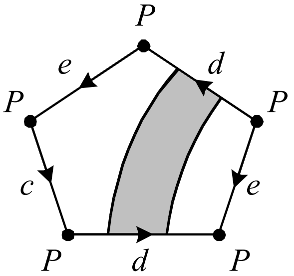
(iii) $V = 1$, $E = 5$, $F = 2$; hence $\chi = -2$.

There are no boundary edges so $\beta = 0$.

The surface is non-orientable since it contains a Möbuis band (shown).

Hence the surface is $4\mathbb{R}P^2$.





10. (i) $\sum(\text{vertex degrees}) = 2E \;\Rightarrow\; kV = 2E$

(ii) $\sum(\text{face degrees}) = 2E \;\Rightarrow\; 6F = 2E \;\Rightarrow\; 6F = kV$.

(iii) $\chi = V - E + F = \dfrac{6F}{k} - 3F + F \;\Rightarrow\; \chi = F\left(\dfrac{6}{k} - 2\right)$.

(iv) (a) $\chi = 2 \;\Rightarrow\; 2 = F\left(\dfrac{6}{k} - 2\right) \;\Rightarrow\; 1 = F\left(\dfrac{3}{k} - 1\right) \;\Rightarrow\; k \le 3$.

$k = 1 \;\Rightarrow\; F = \tfrac{1}{2}$, which is not allowed;

$k = 3 \;\Rightarrow\; 1 = 0 \times F$ which is impossible.

Hence $k = 2,\; F = 2,\; E = 6,\; V = 6$.

(b) $\chi = 1 \;\Rightarrow\; 1 = F\left(\dfrac{6}{k} - 2\right) \;\Rightarrow\; k \le 6$.

$k = 1 \;\Rightarrow\; F = \tfrac{1}{4}$, which is not allowed;

$k = 3 \;\Rightarrow\; 1 = 0 \times F$ which is impossible;

$k > 3 \;\Rightarrow\; F < 0$, which is not allowed.

Hence $k = 2,\; F = 1,\; E = 3,\; V = 3$.

(c) $\chi = 0 \;\Rightarrow\; k = 3,\; F = n,\; E = 3n,\; V = 2n$ where $n \in \mathbb{Z}^+$.

(d) $\chi = -1 \;\Rightarrow\; -1 = F\left(\dfrac{6}{k} - 2\right) \;\Rightarrow\; 1 = F\left(2 - \dfrac{6}{k}\right) \;\Rightarrow\; k \le 6$.

$k \le 2 \;\Rightarrow\; F < 0$, which is not allowed;

$k = 3 \;\Rightarrow\; 1 = 0 \times F$ which is impossible;

$k = 5 \;\Rightarrow\; F \notin \mathbb{Z}$ which is impossible.

Hence $k = 4,\; F = 2,\; E = 6,\; V = 3$
or $k = 6,\; F = 1,\; E = 3,\; V = 1$.

11. (i) $ab^{-1}cadcb^{-1} = 1 \;\rightarrow\; c\underline{\underline{b^{-1}ab^{-1}}}cad = 1$ (cycling)

$\rightarrow\; ccb\underline{\underline{a^{-1}}}bad = 1$ (lemma 1)

$\rightarrow\; ccbbaad = 1$ (lemma 1)

$\rightarrow\; ccbbaaede^{-1} = 1$ (inserting a cuff)

The surface is $3\mathbb{R}P^2 \# D^2$ so $k = 0,\; m = 3,\; n = 1$.

(ii) $aba^{-1}cdb^{-1}d^{-1}c^{-1}=1 \quad \rightarrow \quad aba^{-1}\underline{\underline{x}}b^{-1}x^{-1}=1$ (where $x=cd$)

$\rightarrow \quad xaba^{-1}b^{-1}x^{-1}=1$ (lemma 2)

$\rightarrow \quad aba^{-1}b^{-1}x^{-1}x=1$ (cycling)

$\rightarrow \quad aba^{-1}b^{-1}=1.$

The surface is T^2 so $k=1$ $m=0$ $n=0$.

(iii) $aba^{-1}c=1,\ \ cded^{-1}=1,\ \ efgf^{-1}=1$

$\rightarrow \quad aba^{-1}c=1,\ \ cded^{-1}=1,\ \ e=fg^{-1}f^{-1}$

$\rightarrow \quad aba^{-1}c=1,\ \ cdfg^{-1}f^{-1}d^{-1}=1$

$\rightarrow \quad aba^{-1}c=1,\ \ c=dfgf^{-1}d^{-1}$

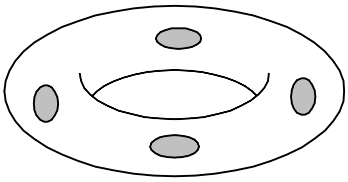
$\rightarrow \quad aba^{-1}dfgf^{-1}d^{-1}=1$

$\rightarrow \quad aba^{-1}xgx^{-1}=1$ (where $x=df$).

The surface is $2D^2$ so $k=0$, $m=0$ $n=2$.

12. (i)
$\beta=0$ $3T^2$
$\beta=2$ $2T^2 \# 2D^2$
$\beta=4$ $T^2 \# 4D^2$
$\beta=6$ $6D^2$

(ii)



(iii)
$\beta=0$ $7\mathbb{R}P^2$
$\beta=1$ $6\mathbb{R}P^2 \# D^2$
$\beta=2$ $5\mathbb{R}P^2 \# 2D^2$
$\beta=3$ $4\mathbb{R}P^2 \# 3D^2$
$\beta=4$ $3\mathbb{R}P^2 \# 4D^2$
$\beta=5$ $2\mathbb{R}P^2 \# 5D^2$
$\beta=6$ $\mathbb{R}P^2 \# 6D^2$

(iv) $3T^2 \# \mathbb{R}P^2 \cong 7\mathbb{R}P^2$
$2T^2 \# 2D^2 \# \mathbb{R}P^2 \cong 5\mathbb{R}P^2 \# 2D^2$
$T^2 \# 4D^2 \# \mathbb{R}P^2 \cong 3\mathbb{R}P^2 \# 4D^2$
$6D^2 \# \mathbb{R}P^2$

(v) No. The surfaces of the form $S \# \mathbb{R}P^2$ where S is orientable with $\chi=-4$ are listed in part (iv). The list does not include, for example, $6\mathbb{R}P^2 \# D^2$ which is non-orientable with $\chi=-5$.

13. (i) (a) Solving for w in terms of z gives:

$$w^2+(2z-3)w+(2z^2-z+3)=0$$

$$\Rightarrow \quad w=\frac{-(2z-3)\pm\sqrt{(2z-3)^2-4(2z^2-z+3)}}{2}$$

$$\Rightarrow \quad =\frac{(3-2z)\pm\sqrt{-4z^2-8z-3}}{2}.$$

There is a single solution for w

$$\Leftrightarrow \quad 4z^2+8z+3=0$$

$$\Leftrightarrow \quad (2z+1)(2z+3)=0$$

$$\Leftrightarrow \quad z=-\tfrac{1}{2},-\tfrac{3}{2}.$$

There are finite branch points at $\left(-\frac{1}{2},2\right)$ and $\left(-\frac{3}{2},3\right)$.

Hence there are no infinite branch points.

(b) Solving for w in terms of z gives:

$$w^2+(4z-2)w+(4z^2-z+3)=0$$

$$\Rightarrow \quad w=\frac{-(4z-2)\pm\sqrt{(4z-2)^2-4(4z^2-z+3)}}{2}$$

$$\Rightarrow \quad =(1-2z)\pm\sqrt{-3z-2}.$$

There is a single solution for $w \Leftrightarrow 3z+2=0 \Leftrightarrow z=-\frac{2}{3}$.

There is a finite branch point at $\left(-\frac{2}{3},\frac{7}{3}\right)$.

Hence (∞,∞) is also a branch point.

(ii) (a) (∞, w) lies on the conic $\Leftrightarrow$ $(0, w)$ lies on

$2+2\tilde{z}w+\tilde{z}^2w^2-\tilde{z}-3\tilde{z}^2w+3\tilde{z}^2=0$ which it does not.

(∞,∞) lies on the conic $\Leftrightarrow$ $(0, 0)$ lies on

$2\tilde{w}^2+2\tilde{z}\tilde{w}+\tilde{z}^2-\tilde{z}\tilde{w}^2-3\tilde{z}^2\tilde{w}+3\tilde{z}^2\tilde{w}^2=0$ which it does.

There are two finite branch points and a single point lying over $z=\infty$ so (∞,∞) is a pinch point.

The conic is of type (c).

(b) The conic has a single finite branch point so (∞,∞) is also a branch point.

The conic is of type (b) where one of the branch points is (∞,∞).

14. (i) Since $\beta(2T^2)=0=\beta(k\mathbb{R}P^2)$, the only restriction is on the Euler characteristic. We have:

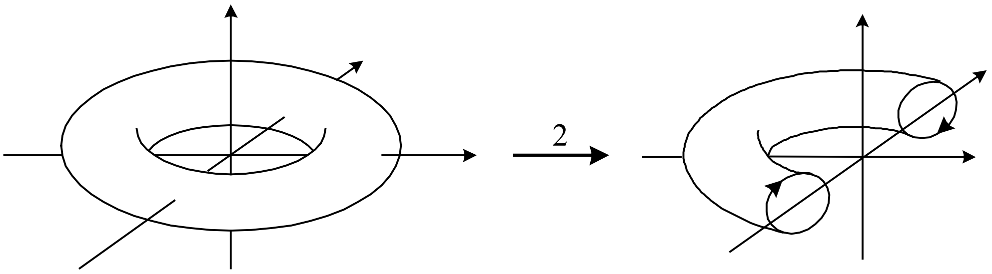
$$\chi(2T^2) = 2\chi(k\mathbb{R}P^2) \;\Rightarrow\; -2 = 2(2-k) \;\Rightarrow\; k = 3.$$

(ii) $\chi(mT^2) = 2\chi(k\mathbb{R}P^2) \qquad \Rightarrow \qquad 2-2m = 2(2-k)$

$\Rightarrow \qquad 1-m = 2-k$

$\Rightarrow \qquad k = m+1.$

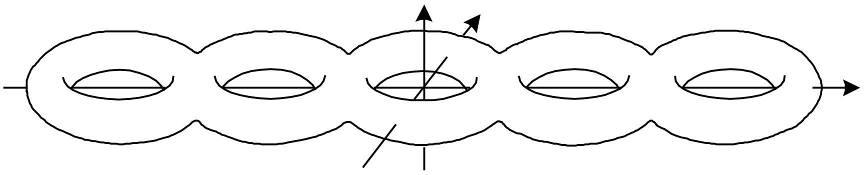
(iii) Identifying points under the mapping $T^2 \to T^2$, $\mathbf{x} \mapsto -\mathbf{x}$ gives a 2-fold unbranched cover $T^2 \xrightarrow{2} 2\mathbb{R}P^2 \cong K^2$ as shown in the diagram below.



(iv) Align mT^2 along the x_1-axis, symmetrically situated with respect to the origin and identify points under the mapping $\mathbf{x} \mapsto -\mathbf{x}$. There are two cases.

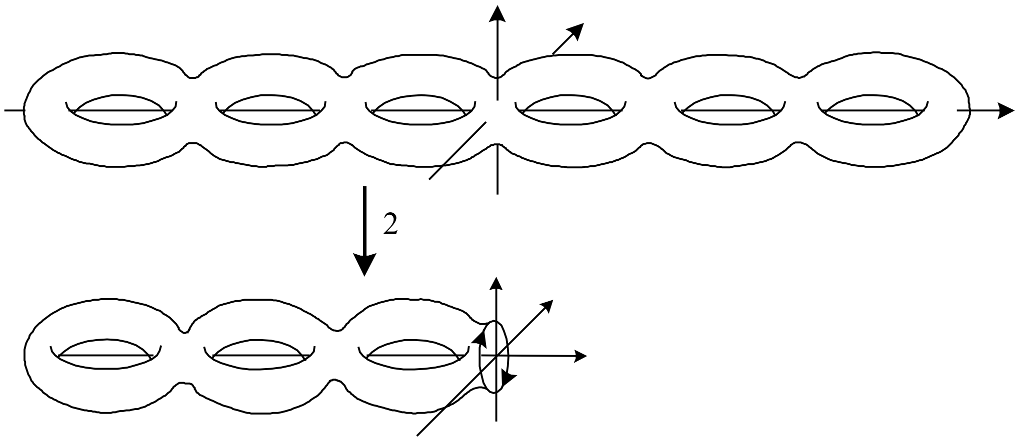
***m* odd**

If $m = 2n + 1$ then the base space is $2\mathbb{R}P^2 \# nT^2 \cong (2n+2)\mathbb{R}P^2$. See the diagram below which is 'part (iii) with handles added'.



***m* even**

If $m = 2n$ then the base space is $\mathbb{R}P^2 \# nT^2 \cong (2n+1)\mathbb{R}P^2$. The diagram is as follows (illustrating the case $n = 3$).



In both cases we obtain a 2-fold covering $(k+1)T^2 \xrightarrow{2} k\mathbb{R}P^2$.

15. (i) Let $f(z) = z^5$, $g(z) = z + 9$.

On C_2 $|g(z)| \leq |z| + 9 = 11 < 32 = |f(z)|$ so, by Rouché's theorem, $V_{C_2}(F) = V_{C_2}(f) = 5 \times 2\pi$. Therefore all 5 zeros of F lies inside C_2.

(ii) Let $h(z) = z^5 + z$, $k(z) = 9$.

On C_1 $|h(z)| \leq |z^5| + |z| = 2 < 9 = |k(z)|$ so, by Rouché's theorem, $V_{C_2}(F) = V_{C_2}(k) = 0$. Therefore F has no zeros inside C_1.

(iii) If $z = x \in \mathbb{R}$, $F'(x) = 4x^4 + 1 > 0$; therefore F is increasing on $\mathbb{R}$ so F has exactly one real zero.

$F(-2) = -32 + 2 + 9 < 0$, $F(-1) = -1 - 1 + 9 > 0$ so the real zero lies in the interval $]-2, -1[$.

(iv) $F(iy) = 9 + i(y^5 + y) \neq 0$ for all $y \in \mathbb{R}$. Therefore F has no zeros on the imaginary axis.

(v) On b: $F(iy) = 9 + i(y^5 + y)$ so $\operatorname{Re}(F(iy)) = 9$, $\operatorname{Im}(F(iy)) > 0$,

$F(2i) = 9 + 34i$, $F(0) = 9$. Therefore $-\frac{\pi}{2} < V_b(F) < 0$.

On a: $z \in \mathbb{R} \Rightarrow F(z) \in \mathbb{R} \Rightarrow V_a(F) = 0$.

On C: From part (i), using the extension to Rouché's theorem,

$$V_C(F) = V_C(f) + \varepsilon = 5 \times \frac{\pi}{2} + \varepsilon, \quad \text{where } |\varepsilon| < \pi.$$

Hence $\frac{3\pi}{2} < V_C(F) < \frac{5\pi}{2}$.

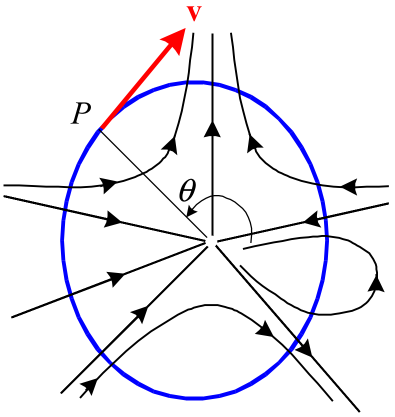
Let D be the closed curve $a + b + C$. Then $\pi < V_D(F) < 7\frac{5\pi}{2}$.

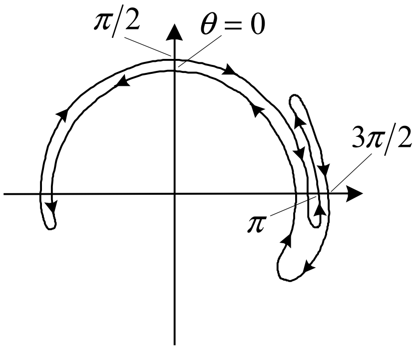
Since $V_D(F)$ is a multiple of 2π, $V_D(F) = 2\pi$, so F has one zero inside D.

(vi) Since F has real coefficients, its non-real zeros are in complex conjugate pairs. Therefore F has one zeros in the each quadrant together with a real zero in $]-2, -1[$.

16. (i) In each case, consider a loop l which winds once round the rest point in an anti-clockwise direction. As P traces out l, the 'sharp ends' of the translated vectors trace our a curve whose winding number equals the index of the rest point.

(a)

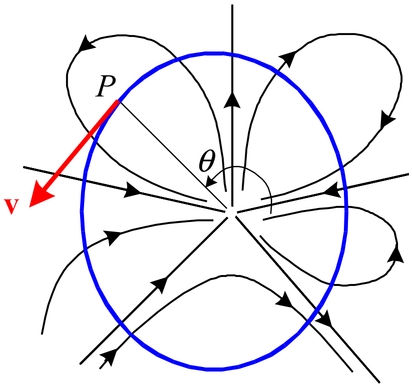


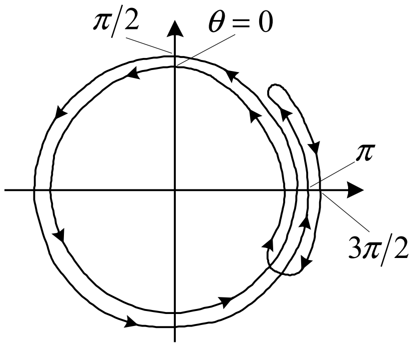


winding number = 0

The rest point has index 0.

(b)

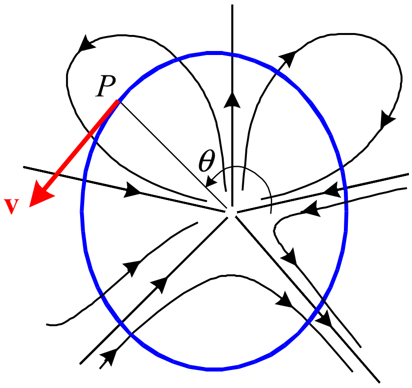


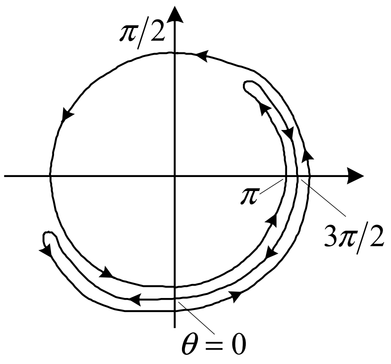


winding number = 2

The rest point has index 2.

(c)





winding number = 1

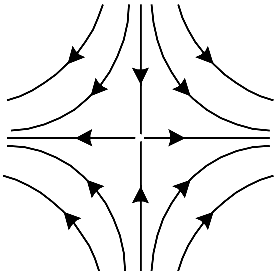
The rest point has index 1.

(ii) Index sum = 0 + 2 = 2. By the Poincaré index theorem the surface has $\chi = 2$ and is therefore a sphere.

(iii) Index sum = 0. By the Poincaré index theorem the surface has $\chi = 0$ and is therefore a torus.

(iv) Index sum = 1. By the Poincaré index theorem the surface has $\chi = 1$. Since no orientable surface without boundary has $\chi = 1$ there is no such surface which admits a flow with only one singularity of type (c) and no others.

(v)



(vi) Index sum = 2 × 1 + 4 × (–1) = –2. By the Poincaré index theorem the surface has $\chi = -2$ and is therefore a double torus.

The following diagram shows such a flow.