M435/R

The Open University

Fourth Level Course Examination 2001

Metric and Topological Spaces and Geometric Topology

Wednesday 17 October 2001 10.00 am – 1.00 pm

Time allowed: 3 hours

This paper is in two Parts. Part I (Questions 1–8) is on Metric and Topological Spaces and Part II (Questions 9–16) is on Geometric Topology.

Your examination grade will be the sum of your best **NINE** question scores, where *not more than six* of these come from a single Part.

If you exceed six questions from one Part, all questions will be marked but credit will be given only for your best **NINE** questions within the restriction stated above. You may cross out any work that you do not wish the examiner to mark.

Use a **separate** answer book for **each** part.

All questions carry equal marks. The allocation of marks within a question is indicated by a number in brackets, [ ], beside the question.

At the end of the examination

Check that you have written your personal identifier and examination number on each answer book used. **Failure to do so will mean that your work cannot be identified.** Attach all your answer books together using the fastener provided.

The use of calculators is not permitted in this examination

Answer as many questions as you wish. Full marks may be obtained by **complete** *answers to* **NINE** *questions, provided that no more than* **SIX** *questions have been selected from any one part. All questions carry equal marks.*

PART I METRIC AND TOPOLOGICAL SPACES

Question 1

(i) Let $A = \{a, b, c, d\}$, $B = \{1, 2, 3, 4\}$ and let $f : A \longrightarrow B$ be given by

$$f(a) = 2, \quad f(b) = f(c) = 3, \quad f(d) = 4.$$

Write down:

(a) $f(f^{-1}(\{1, 2, 3\}))$;

(b) $f(f^{-1}(\{2, 3, 4\}))$. [4]

(ii) Let $g : C \longrightarrow D$ be a function, where C and D are non-empty sets, and let E be a subset of D.

(a) Prove that

$$g(g^{-1}(E)) \subset E.$$ [3]

(b) Prove that if $E \subset g(C)$ then

$$E = g(g^{-1}(E)).$$ [4]

Question 2

Let $d : \mathbf{R} \times \mathbf{R} \longrightarrow \mathbf{R}$ be defined by

$$d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ |x - 1| + |y - 1|, & \text{if } x \neq y. \end{cases}$$

(i) Show that d is a metric on $\mathbf{R}$. [5]

(ii) Determine the following open balls with respect to the metric d:

(a) $B_1(0)$;

(b) $B_\delta(1)$, where δ is a positive real number. [3]

(iii) Show that the function $f : \mathbf{R} \longrightarrow \mathbf{R}$ defined by

$$f(x) = 1 - x, \ x \in \mathbb{R},$$

is not (d, d)-continuous at 1. [3]

Question 3

Let $\mathscr{T}$ be the collection of subsets of the set $\mathbf{Z}$ of all integers defined as follows: $V \in \mathscr{T}$ if either $0 \notin V$ or $E \subset V$, where $E = \{\ldots, -2, 0, 2, 4, \ldots\}$ is the set of all even integers.

(i) Show that $\mathscr{T}$ is a topology on $\mathbf{Z}$. [7]

(ii) Determine the topology induced by $\mathscr{T}$ on the set $\{-1, 0, 2\}$. [4]

Question 4

Let $T_1 = \{A, \mathscr{T}\}$ where $A = \{a, b, c\}$ and $\mathscr{T}$ is the topology

$$\{\varnothing, \{b\}, \{a,b\}, \{b,c\}, A\}$$

and let $T_2 = \{\mathbf{Z}, \mathscr{V}\}$ where $\mathbf{Z}$ is the set of all integers and $\mathscr{V}$ is the topology defined as follows: $V \in \mathscr{V}$ if either $0 \notin V$ or $1 \in V$.

[You need **not** check that $\mathscr{T}$, $\mathscr{V}$ are topologies.]

(i) Prove that $f : A \longrightarrow \mathbf{Z}$ given by

$$f(a) = f(b) = 1, \quad f(c) = 0$$

is $(\mathscr{T}, \mathscr{V})$-continuous. [4]

(ii) Prove that $g : A \longrightarrow \mathbf{Z}$ given by

$$g(a) = g(c) = 1, \quad g(b) = 2$$

is not $(\mathscr{T}, \mathscr{V})$-continuous. [3]

(iii) State whether or not $h : A \longrightarrow \mathbf{Z}$ given by

$$h(a) = h(c) = 0, \quad h(b) = 1$$

is $(\mathscr{T}, \mathscr{V})$-continuous, justifying your answer. [4]

Question 5

Let $\mathscr{T}$ be the topology on $\mathbf{R}$ which consists of all sets V such that either $0 \notin V$ or there exists a positive real number r such that $(-r, r) \subset V$, and put $T = \{\mathbf{R}, \mathscr{T}\}$.

[You need **not** verify that $\mathscr{T}$ is a topology on $\mathbf{R}$.]

(i) Describe the closed sets of the topological space T. [3]

(ii) For each of the following subsets of $\mathbf{R}$, state whether or not it is a closed set of T, justifying your answer:

(a) $(-1, 1)$;

(b) $\{-1, 1\}$;

(c) $\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\}$. [6]

(iii) For any set in part (ii) which is not closed in T, write down its closure in T. [2]

Question 6

(i) For each positive integer n, let

$$A_n = \left\{(x_1, x_2) \in \mathbf{R}^2 : \ x_1^2 + x_2^2 = \frac{1}{n}\right\}$$

and let

$$A = \bigcup_{n=1}^{\infty} A_n, \quad B = A \cup \{(0,0)\}.$$

For each of the subsets A and B of $\mathbf{R}^2$ with the usual topology, state whether or not it is compact, briefly justifying your answer. [3]

(ii) Let $\mathscr{T}_1$ be the topology on the set $\mathbf{N}$ of positive integers consisting of $\varnothing$ and the sets

$$\{n \in \mathbf{N} : n \geqslant k\}$$

for each positive integer k, and let $T_1 = \{N, \mathscr{T}_1\}$. [You need **not** verify that $\mathscr{T}_1$ is a topology.] Prove that T_1 is a compact space. [4]

(iii) Let $\mathscr{T}_2$ be the topology on the set $\mathbf{N}$ of positive integers consisting of $\varnothing, \mathbf{N}$ and the sets

$$\{n \in \mathbf{N} : \ n \leqslant k\}$$

for each positive integer k, and let $T_2 = \{\mathbf{N}, \mathscr{T}_2\}$. [You need **not** verify that $\mathscr{T}_2$ is a topology.] State whether or not T_2 is a compact space and justify your answer. [4]

Question 7

(i) Prove that if H is a connected subspace of a topological space and $H \subset K \subset Cl(H)$, then K is connected. [5]

(ii) For each positive integer n, let L_n be the line segment in $\mathbf{R}^2$ joining $\left(1, \frac{1}{n}\right)$ and $\left(-1, -\frac{1}{n}\right)$ and let

$$K = \left\{\left(-\frac{1}{2}, 0\right), \left(\frac{1}{2}, 0\right)\right\} \cup \bigcup_{n=1}^{\infty} L_n.$$

Prove that the subspace K of $\mathbf{R}^2$ with the usual topology is connected. [6]

Question 8

Let d be the metric on the set $\mathbf{N}$ of positive integers defined by

$$d(i,j) = \left|\frac{1}{i} - \frac{1}{j}\right|, \quad i, j, \in \mathbf{N}.$$

[You need **not** verify that d is a metric.]

(i) Prove that if $x_n = n$ for each positive integer n, then (x_n) is a Cauchy sequence in the metric space $\{\mathbf{N}, d\}$. [4]

(ii) Show that if $k \in \mathbf{N}$ then

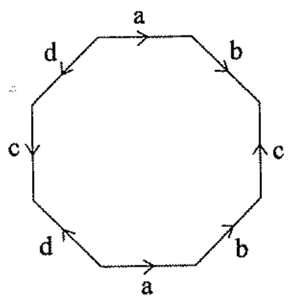
$$d(k, x_n) \geqslant \frac{1}{2k}$$

for each integer n such that $n \geqslant 2k$. [3]

(iii) Deduce that the metric space $\{\mathbf{N}, d\}$ is not complete. [4]

Question 9

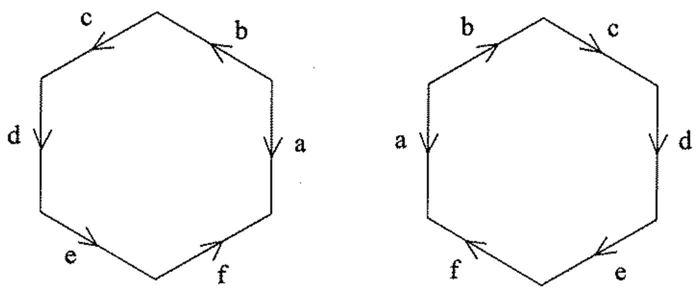
(i) Draw a copy of the polygon below on your answer sheet and, by making the identifications shown, label each vertex. Hence determine χ (the Euler characteristic) and β (the number of boundary components) for the corresponding surface, and explain why it is not orientable by drawing a Möbius band on your copy of the figure. Hence write down the surface as a connected sum of copies of $\mathbb{RP}^2$ and D^2.



[3]

(ii) By changing the direction of exactly one edge, obtain an orientable surface, and draw this on your answer sheet. Hence, by making the identifications shown on your new sketch, label each vertex, and for this new surface determine χ (the Euler characteristic) and β (the number of boundary components). Hence write down the surface as a connected sum of copies of T^2 and D^2. [4]

(iii) Draw a copy of the polygons below on your answer sheet and, by making the identifications shown, label each vertex. Hence determine χ (the Euler characteristic) and β (the number of boundary components) for the corresponding surface, and determine if it is orientable or not. Hence write down the surface as a connected sum of copies of S^2, T^2, and D^2 or $\mathbb{RP}^2$ and D^2.



[4]

Question 10

(i) Write down the condition that a closed surface subdivided into F 7-sided polygons has k faces meeting at each vertex. [1]

(ii) Write down a formula connecting F, k, and the Euler characteristic χ. [2]

(iii) Hence find all possible regular subdivisions of this type when

(a) $\chi = -2$; [4]

(b) $\chi = -3$; [4]

Question 11

Reduce each of the following edge equations (or sets of edge equations) to canonical form. Hence or otherwise classify the surface they define as a connected sum of k copies of the torus T^2, m copies of the real projective plane $\mathbb{RP}^2$, and n copies of the disc D^2, for values of k, m, and n which you should state.

(i) $a^{-1}cabdb^{-1}c^{-1}d^{-1} = 1$ [3]

(ii) $a^{-1}cab^{-1}db^{-1}c^{-1}d = 1$ [4]

(iii) $aba^{-1} = 1,\ cbec^{-1} = 1,\ def = 1,\ fd^{-1} = 1$ [4]

Question 12

In this question you are asked to investigate all the surface that can be formed from a 6-sided polygon with three pairs of identified edges.

(i) Write down a formula for the Euler characteristic χ of such a surface in terms of V, the number of vertices, E, the number of edges, and F the number of faces, and deduce that χ must be greater than -2. [2]

(ii) Write down in connected sum form all the possible surfaces that can be made in this way, and explain why there are no other possibilities. [4]

(iii) Write down an edge equation for each of the surfaces you listed in (ii) which shows that they can be formed from a 6-sided polygon with three pairs of identified edges. [5]

Question 13

(i) For each of the following conics (which you may assume are non-degenerate) locate the branch points when they exist.

(a) $6z^2 - 4zw + w^2 - 11z + 5 = 0$; [4]

(b) $z^2 + 6zw + 9w^2 - z - 1 = 0$; [3]

(ii) Hence categorise each of them as one of the following:

(a) homeomorphic to the z-sphere;

(b) a double cover of the z-sphere branched over two points;

(c) a double cover of the z-sphere branched over two points and having a pinch point. [4]

Question 14

(i) A surface S with edge equation $ababcdc^{-1} = 1$ has a three-fold cover by a surface S' given by the edge equations

$$a_1b_1a_1b_3c_1d_1c_2^{-1} = 1$$
$$a_2b_2a_3b_2c_2d_2c_3^{-1} = 1$$
$$a_3b_3a_2b_1c_3d_3c_1^{-1} = 1$$

Use the method of inserting vertices to determine the Euler characteristic χ and the number of boundary components β of the surface S'. [6]

(ii) Determine if the surface S' is orientable or non-orientable. [2]

(iii) Hence write the surface S' in connected sum form. [3]

Question 15

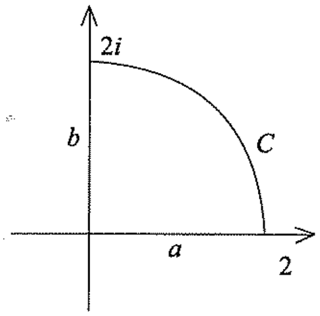
(i) Show that all four zeros of $F(z) = z^4 + z + 7 = 0$ lie inside the circle $C_2 = \{z : |z| = 2\}$. [1]

(ii) Show that $F(z)$ has no zeros inside the circle $C_1 = \{z : |z| = 1\}$. [1]

(iii) Show that $F(z)$ has no real zeros. [1]

(iv) Show that $F(z)$ has no zeros on the imaginary axis. [1]

(v) By considering the image under F of the contour below, find out how many zeros of $F(z)$ have both real and imaginary parts positive.



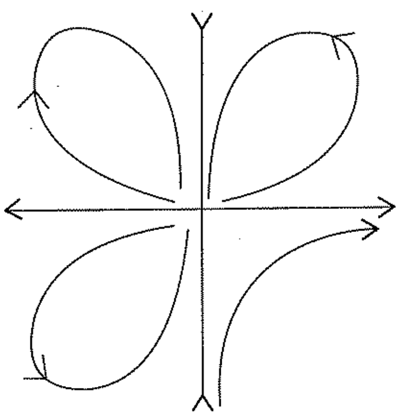
(The contour consists of a, the real axis from 0 to 2, followed by C, the quarter-circle radius 2 centre the origin, from 2 to $2i$, followed by b, the imaginary axis from $2i$ to 0.) [6]

(vi) Hence determine the number of zeros of $F(z)$ in each quadrant. [1]

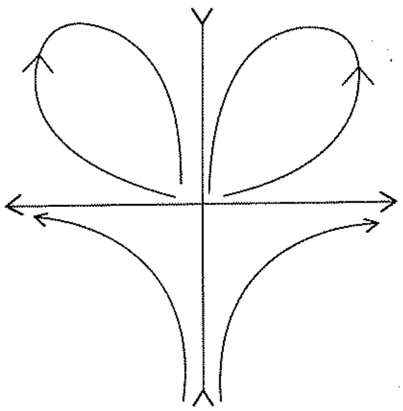
Question 16

(i) Calculate the indices of the rest points in the diagrams (a), (b), and (c) below.

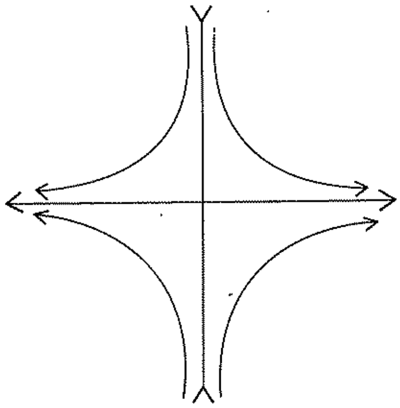
(a)



(b)



(c)



[3]

(ii) On what closed orientable surface, if any, could there be a flow with precisely 1 singularity of type (a), 2 singularities of type (c), and no others? [2]

(iii) On what closed orientable surface, if any, could there be a flow with precisely 1 singularity of type (b), and no others? [2]

(iv) On what closed orientable surface, if any, could there be a flow with precisely 1 singularity of type (a), one of type (b) and five of type (c), and no others? [2]

(v) There is exactly one closed orientable surface which could carry a flow with the following: one singularity of type (a) and a singularity of index 0. Which surface is it? Illustrate by a sketch that such a flow is possible. [2]

[END OF QUESTION PAPER]