Metric and Topological Spaces

1. (i) (a) $f^{-1}\left(f(\{b,c\})\right) = f^{-1}(\{1,2\}) = \{a,b,c\}$.

   (b) $f^{-1}\left(f(\{c,d\})\right) = f^{-1}(\{2,3\}) = \{c,d\}$.

   (ii) (a) Let $x \in E$. Then $f(x) \in f(E)$ so $x \in f^{-1}\left(f(E)\right)$.

   Therefore $E \subset f^{-1}\left(f(E)\right)$.

   (b) Suppose that f is injective and let $x \in f^{-1}\left(f(E)\right)$.

   Then $f(x) \in f(E)$ so there exists $y \in E$ such that $f(x) = f(y)$.

   Since f is injective, this implies $x = y$, so $x \in E$.

   Therefore $f^{-1}\left(f(E)\right) \subset E$.

2. (i) We need to verify the conditions (M1), (M2) and (M3).

   (M1) For all $x, y \in \mathbb{R}$, $d(x,y) \geq 0$ (since $|a| \geq 0$ for all $a \in \mathbb{R}$).

   If $x \neq y$ then $d(x,y) \geq |x-y| > 0$.

   Therefore (M1) is satisfied.

   (M2) If $x \neq y$ then $d(x,y) = |x| + |y| + |x-y| = |y| + |x| + |y-x| = d(y,x)$

   Therefore (M2) is satisfied.

   (M3) Let x, y and z be *distinct* elements of $\mathbb{R}$. Then

   $$\begin{aligned} &d(x,y) + d(y,z) \\ &\quad = |x| + |y| + |x-y| + |y| + |z| + |y-z| \\ &\quad \geq |x| + |z| + |x-y| + |y-z| \qquad \left(\text{since } |y| \geq 0\right) \\ &\quad \geq |x| + |z| + |x-y+y-z| \qquad \left(\text{since } |a| + |b| \geq |a-b| \text{ for all } a,b \in \mathbb{R}\right) \\ &\quad = |x| + |z| + |x-z| \\ &\quad = d(x,z). \end{aligned}$$

   Therefore (M3) is satisfied.

   Hence d is a metric on $\mathbb{R}$.

   (ii) (a)
   $$\begin{aligned} B_1(0) &= \{x \in \mathbb{R} : d(x,0) < 1\} \\ &= \{0\} \cup \{x \in \mathbb{R} : |x| + |0| + |x-0| < 1\} \\ &= \{0\} \cup \{x \in \mathbb{R} : 2|x| < 1\} = \left(-\tfrac{1}{2}, \tfrac{1}{2}\right). \end{aligned}$$

(b) $B_1(1) = \{x \in \mathbb{R}: d(x,1) < 1\}$

$= \{1\} \cup \{x \in \mathbb{R}: |x| + |1| + |x-1| < 1\}$

$= \{1\} \cup \{x \in \mathbb{R}: |x| + |x-1| < 0\}$

$= \{1\} \cup \varnothing$

$= \{1\}.$

(iii) Let $\varepsilon = 1$. Then $B_\varepsilon(f(0)) = B_1(1) = \{1\}$, by part (ii)(b).

For all $\delta > 0$, then, using a similar argument to part (ii)(a),

$$B_\delta(0) = \{0\} \cup \{x \in \mathbb{R}: 2|x| < \delta\} = \left(-\tfrac{1}{2}\delta, \tfrac{1}{2}\delta\right).$$

Therefore $\quad f(B_\delta(0)) = \left(1 - \tfrac{1}{2}\delta,\ 1 + \tfrac{1}{2}\delta\right) \not\subset \{1\}$.

Thus, for all $\delta > 0$, $f(B_\delta(0)) \not\subset B_\varepsilon(f(0))$, so f is not (d, d)-continuous at 0.

3. (i) We need to verify the conditions (T1), (T2), (T3).

(T1) $1 \notin \varnothing$ so $\varnothing \in \mathscr{T}$ and $\mathbb{O} \subset \mathbb{N}$ so $\mathbb{N} \subset \mathscr{T}$. Therefore (T1) is satisfied.

(T2) Let $U, V \in \mathscr{T}$.
If $1 \notin U$ or $1 \notin V$ then $1 \notin U \cap V$ so $U \cap V \in \mathscr{T}$;
otherwise $\mathbb{O} \subset U$ and $\mathbb{O} \subset U$ so $\mathbb{O} \subset U \cap V$; hence $U \cap V \in \mathscr{T}$.
Therefore (T2) is satisfied.

(T3) Let $\{U_i\}_{i \in I}$ be a family of sets from $\mathscr{T}$.
If $1 \notin U_i$ for all $i \in I$ then $1 \notin \bigcup_{i \in I} U_i$; hence $\bigcup_{i \in I} U_i \in \mathscr{T}$.
Otherwise $\mathbb{O} \subset U_j$ for some $j \in I$ so $\mathbb{O} \subset U_j \subset \bigcup_{i \in I} U_i$;
hence $\bigcup_{i \in I} U_i \in \mathscr{T}$. Therefore (T3) is satisfied.

Since $\mathscr{T}$ satisfies (T1), (T2), (T3), it is a topology on $\mathbb{N}$.

(ii) Let $A = \{1, 2, 3\}$ and let $\mathscr{T}_A$ denote the induced topology.
Now $\{2\} = \{2\} \cap A$ and $\{2\} \in \mathscr{T}$, so $\{2\} \in \mathscr{T}_A$.
Similarly, $\{3\} = \{3\} \cap A$ and $\{3\} \in \mathscr{T}$, so $\{3\} \in \mathscr{T}_A$.

If $1 \in U$ then $\mathbb{O} \subset U$ so $\{1, 3\} \subset U \cap A$; hence an open set of $\mathscr{T}_A$ containing 1 must also contain 3.
Therefore $\mathscr{T}_A = \{\varnothing,\ \{2\},\ \{3\},\ \{2,3\},\ \{1,3\},\ A\}$.

4. (i) Let V be an open set in T_2.

If $0 \notin V$ then $f^{-1}(V) = \begin{cases} \varnothing & \text{if } 1 \notin V \\ \{a, b\} & \text{if } 1 \in V, \end{cases}$ so $f^{-1}(V)$ is open in T_1.

If $1 \in V$ then $f^{-1}(V) = \begin{cases} \{a, b\} & \text{if } 0 \notin V \\ A & \text{if } 0 \in V, \end{cases}$ so $f^{-1}(V)$ is open in T_1.

Therefore f is $(\mathscr{T}, \mathscr{V})$-continuous.

(ii) $\{1\} \in V$ but $g^{-1}(\{1\}) = \{b, c\} \notin \mathscr{T}$. Therefore g is not $(\mathscr{T}, \mathscr{V})$-continuous.

(iii) $\{0, 1\} \in V$ but $h^{-1}(\{0, 1\}) = \{b, c\} \notin \mathscr{T}$. Therefore h is not $(\mathscr{T}, \mathscr{V})$-continuous.

5. $B_1 = \{1, 2, 4, 6, \ldots\}$, $B_2 = \{2\}$,
$B_3 = \{3, 4, 6, 8, \ldots\}$, $B_4 = \{4\}$,
$B_5 = \{5, 6, 8, 10, \ldots\}$, $B_6 = \{6\}, \ldots$

(i) For each $n \in \mathbb{N}$, $n \in B_n$. Therefore $\mathbb{N} = \bigcup_{n \in \mathbb{N}} B_n$ so (B1) is satisfied.

Let $k \in B_n \cap B_m$ where $n \neq m$. Then k is even.
(If k is odd then $k \notin B_r$ for $k \neq r$.)

Hence $B_k = \{k\}$ so $k \in B_k \subset B_n \cap B_m$; therefore (B2)′ is satisfied.

Since (B1) and (B2)′ are satisfied, $\mathscr{B}$ is a synthetic basis for $\mathbb{N}$.

(ii) Let $\mathbb{E} = \{\text{even positive integers}\} = \{2, 4, 6, 8, \ldots\}$.

For each $n \in \mathbb{E}$, $B_n = \{n\}$, so $\mathbb{E} = \bigcup_{n \in \mathbb{E}} B_n$.

Therefore $\mathbb{E}$ is open, so $\mathbb{O} = \mathbb{N} - \mathbb{E}$ is closed.

(iii) (a) Let U be an open set containing 3. Since B_3 is the only basic open set containing 3, $B_3 \subset U$. Therefore $4 \in B_3 \cap \{2, 4\} \subset U \cap \{2, 4\}$.
Thus every open set containing 3 meets $\{2, 4\}$ so 3 is a limit point of $\{2, 4\}$.

(b) $\text{Cl}(\{2, 4\}) = \{1, 2, 3, 4\}$.

(No explanation need be given here, but here it is:
1 is a limit point of $\{2, 4\}$ by a similar argument to (iii)(a);
$k \geq 5$ is not a limit point of $\{2, 4\}$ since $B_k \cap \{2, 4\} = \varnothing$.)

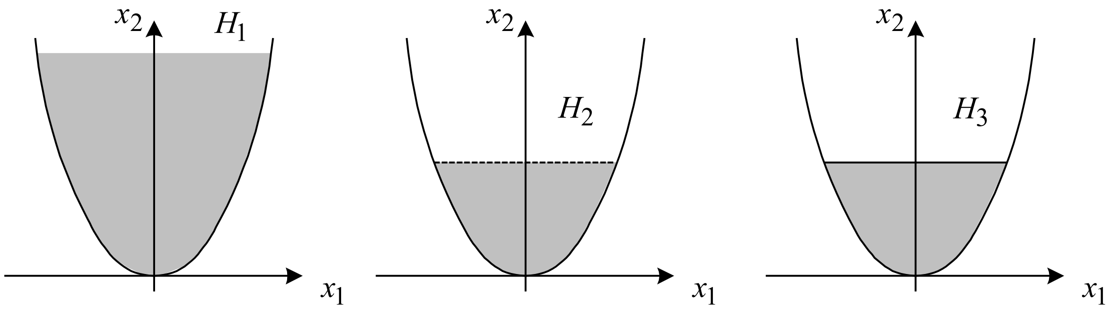
6. (i) Let A be a compact subspace of M. Choose any $a \in M$. Then $\{B_n(a)\}_{n\in\mathbb{N}}$ is an open cover of A (in fact, it is an open cover of M).

Since A is compact, there is a finite subcover of $\{B_n(a)\}_{n\in\mathbb{N}}$, say $\{B_{n_1}(a), B_{n_2}(a), \ldots, B_{n_k}(a)\}$; i.e. $A \subset \left(B_{n_1}(a) \cup B_{n_2}(a) \cup \ldots \cup B_{n_k}(a)\right)$.

Let $N = \max\{n_1, n_2, \ldots, n_k\}$; then $\left(B_{n_1}(a) \cup B_{n_2}(a) \cup \ldots \cup B_{n_k}(a)\right) = B_N(a)$.

Hence $A \subset B_N(a)$ so A is bounded.

(ii)



H_1 is not bounded (for example, $(0, n) \in H_1$ for all positive integers n) so H_1 is not compact.

H_2 is not closed and hence not compact.

H_3 is closed and bounded, hence H_3 is compact.

7. (i) Suppose that H is a connected subspace of T and that $H \subset K \subset \mathrm{Cl}(H)$.

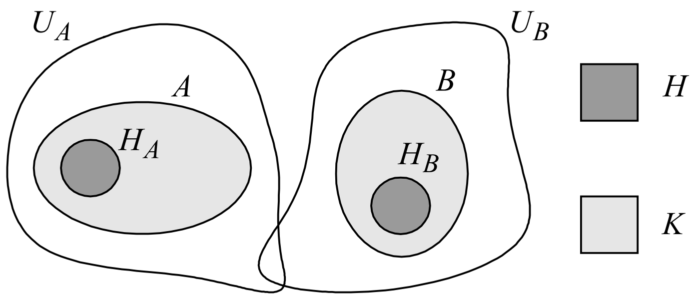
The proof is by contradiction. Suppose K is not connected. Then there exists a partition $A \mid B$ of K. The sets A and B are open in K so there exists sets U_A and U_B, open in T, such that $A = K \cap U_A$, $B = K \cap U_B$.

Let $H_A = A \cap H$ and $H_B = B \cap H$. Since $H \subset K = A \cup B$ it follows that $H = H_A \cup H_B$. Also $H_A = H \cap U_A$ and $H_B = H \cap U_B$ so H_A and H_B are open in H. However H is connected and so does not admit a partition; therefore at least one of H_A and H_B is empty.

Suppose, without loss of generality, that $H_B = \varnothing$. Then $H = H_A \subset A$. Let $b \in B$. Then U_B is an open set containing b which does not meet H, so b is not a limit point of H. Thus $b \in K$ but $b \notin \mathrm{Cl}(H)$ which contradicts $K \subset \mathrm{Cl}(H)$.

Therefore K is connected.

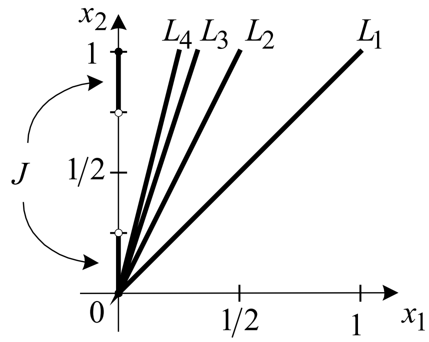
The diagram illustrates the sets involved in the proof. (For an alternative proof, see **S**, page 99.)



(ii) Let $J_1 = \{(x_1, x_2) \in J : x_2 < \frac{1}{4}\}$,

$J_2 = \{(x_1, x_2) \in J : x_2 > \frac{3}{4}\}$

and $H = \bigcup_{n=1}^{\infty} L_n$.



Now J_1 and each L_n is connected (each is homeomorphic to an interval) and, for $n = 1, 2, 3, \ldots$, $J_1 \cap L_n = \{(0, 0)\} \neq \varnothing$.

Therefore $J_1 \cup H$ is connected [Corollary 6.2.16.].

Let $x \in J_2$. Every open set containing x also meets $J_1 \cup H$, so x is a limit point of $J_1 \cup H$. Therefore

$$(J_1 \cup H) \subset J_2 \cup (J_1 \cup H) \subset \text{Cl}(J_1 \cup H); \quad \text{i.e.} \quad (J_1 \cup H) \subset K \subset \text{Cl}(J_1 \cup H).$$

Therefore K is connected (by part (i)).

8. (i) Suppose (x_n) is a convergent sequence in the metric space M. Then (x_n) converges to some $x \in M$.

Let $\varepsilon > 0$. Then $\frac{\varepsilon}{2} > 0$ so there exists a positive integer N such that $d(x_n, x) < \frac{\varepsilon}{2}$ for all $n \geq N$.

Therefore, for all $n, m \geq N$,

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence (x_n) is a Cauchy sequence.

(ii) (a) For all $n, m \in \mathbb{N}$,

$$d(x_n, x_m) = d\left((2n, 1),\ (2m, 1)\right) = \left|\frac{1}{2n} - \frac{1}{2m}\right| + |1 - 1| \leq \frac{1}{2n} + \frac{1}{2m}.$$

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $N > \frac{1}{\varepsilon}$. Then, for all $n, m \geq N$,

$$d(x_n, x_m) \leq \frac{1}{2n} + \frac{1}{2m} \leq \frac{1}{2N} + \frac{1}{2N} = \frac{1}{N} < \varepsilon$$

so (x_n) is a Cauchy sequence.

(b) For all $n, m \in \mathbb{N}$,

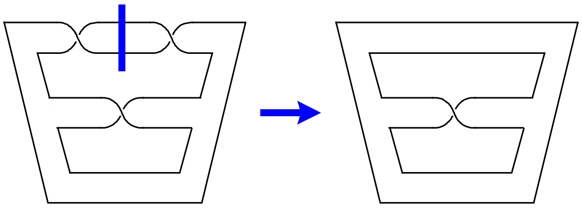
$$d(y_n, y_m) = d\left((1, n),\ (1, m)\right) = \left|\frac{1}{1} - \frac{1}{1}\right| + |n - m| = |n - m|.$$

Let $\varepsilon = \frac{1}{2}$; then for all $n \in \mathbb{N}$, $d(y_{n+1}, y_n) = 1 > \varepsilon$.

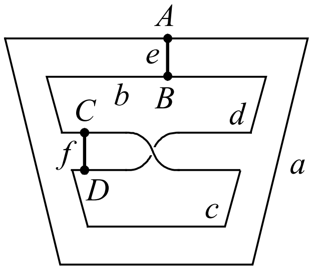
Therefore (y_n) is not a Cauchy sequence.

Part II: Geometric Topology

9. (i) Cut where shown, unwind and re-glue.



The following is a subdivision of the surface.



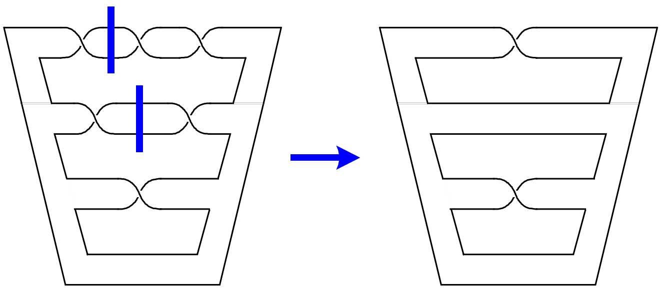
$V = 4$, $E = 6$, $F = 1$. Hence $\chi = -1$.

$\beta = 2$ (the two boundary components are a and bcd).

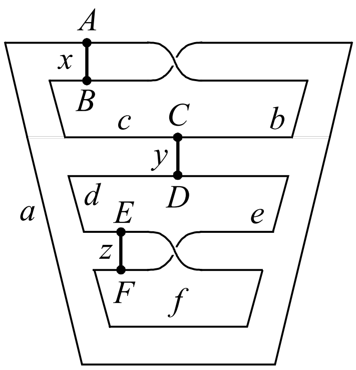
The surface clearly contains a Möbius band and hence is non-orientable.

[The surface is $\mathbb{R}P^2 \# 2D^2$.]

(ii) Make two cuts as shown; for each cut, unwind two half twists and re-glue.



The following is a subdivision of the surface.



$V = 6$, $E = 9$, $F = 1$. Hence $\chi = -2$.

$\beta = 2$ (the two boundary components are acb and def).

Again, the surface clearly contains a Möbius band and so is non-orientable.

[The surface is $2\mathbb{R}P^2 \# 2D^2$.]

10. (i) $kV = 9F$. (Also $2E = 9F$.)

(ii)

$$\chi = -n = V - E + F$$

$$\Rightarrow \quad -n = \frac{9F}{k} - \frac{9F}{2} + F$$

$$\Rightarrow \quad -2kn = F(18 - 7k) \qquad (*)$$

$$\Rightarrow \quad F = \frac{2kn}{7k - 18}.$$

(iii) From part (i), equation (*), $7kF - 2kn = 18F \quad \Rightarrow \quad k = \dfrac{18F}{7F - 2n}$.

(iv) Since k, F and n are positive integers, from part (ii) we have:

$$7k - 18 > 0 \quad \Rightarrow \quad k > \frac{18}{7} \quad \Rightarrow \quad k \geq 3.$$

Now from part (iii), we have:

$$\frac{18F}{7F - 2n} \geq 3 \quad \Rightarrow \quad 18F \geq 21F - 6n \quad \Rightarrow \quad 6n \geq 3F \quad \Rightarrow \quad F \leq 2n.$$

Also, since $2E = 9F$, F is even.

(a) $\boldsymbol{n = 1}$. Then $k = \dfrac{18F}{7F - 2}$ and $F \leq 2$.

$F = 2 \quad \Rightarrow \quad k = \dfrac{36}{12} = 3$. Hence $V = 6$, $E = 9$.

This is the only possible regular subdivision of the given type.

(b) $\boldsymbol{n = 2}$. Then $k = \dfrac{18F}{7F - 4}$ and $F \leq 4$.

$F = 2 \quad \Rightarrow \quad k = \dfrac{36}{10} \notin \mathbb{Z}$. $\qquad F = 4 \quad \Rightarrow \quad k = \dfrac{72}{24} = 3$.

The only possible regular subdivisions of this type has:

$F = 4$, $k = 3$; hence $V = 12$, $E = 18$.

(c) $\boldsymbol{n = 3}$. Then $k = \dfrac{18F}{7F - 6}$ and $F \leq 6$.

$F = 2 \quad \Rightarrow \quad k = \dfrac{36}{8} \notin \mathbb{Z}$. $\qquad F = 4 \quad \Rightarrow \quad k = \dfrac{72}{22} \notin \mathbb{Z}$.

$F = 6 \quad \Rightarrow \quad k = \dfrac{108}{36} = 3$.

The only possible regular subdivisions of this type has:

$F = 6,\ k = 3;$ hence $V = 18,\ E = 27.$

(d) $\boldsymbol{n = 4}$. Then $k = \dfrac{18F}{7F - 8}$ and $F \le 8.$

$F = 2 \ \Rightarrow\ k = \dfrac{36}{6} = 6.$ $\qquad$ $F = 4 \ \Rightarrow\ k = \dfrac{72}{20} \notin \mathbb{Z}.$

$F = 6 \ \Rightarrow\ k = \dfrac{108}{34} \notin \mathbb{Z}.$ $\qquad$ $F = 8 \ \Rightarrow\ k = \dfrac{144}{48} = 3.$

The possible regular subdivisions of this type have:

$F = 2,\ k = 6;$ hence $V = 3,\ E = 9.$

$F = 8,\ k = 3;$ hence $V = 24,\ E = 36.$

[Note: in all these cases, the equations only give what is possible. To show that each of these subdivisions genuinely does exist, we would need to construct them.]

11. (i) $e^{-1} f e^{-1} = 1$

$\rightarrow$ $e^{-1} e^{-1} f = 1$ (cycling)

$\rightarrow$ $e^{-1} e^{-1} x f x^{-1} = 1$ (Step 4: introducing a cuff)

$\rightarrow$ $a a x f x^{-1} = 1$ (relabelling).

Surface is $\mathbb{R}P^2 \# D^2$.

(ii) $a b a^{-1} \underline{\underline{c}} b^{-1} c^{-1} = 1$

$\rightarrow$ $c a b a^{-1} b^{-1} c^{-1} = 1$ (lemma 2)

$\rightarrow$ $a b a^{-1} b^{-1} c^{-1} c = 1$ (cycling)

$\rightarrow$ $a b a^{-1} b^{-1} = 1.$

Surface is T^2.

(iii)
$$aba^{-1}cb^{-1}d=1,\quad def^{-1}ec=1$$
$$\rightarrow \quad aba^{-1}cb^{-1}d=1,\quad d=c^{-1}e^{-1}fe^{-1}$$
$$\rightarrow \quad ba^{-1}cb^{-1}c^{-1}e^{-1}fe^{-1}=1$$
$$\rightarrow \quad e^{-1}\,\underline{\underline{f}}\,e^{-1}aba^{-1}cb^{-1}c^{-1}=1 \qquad \text{(cycling)}$$
$$\rightarrow \quad f^{-1}e^{-1}e^{-1}aba^{-1}cb^{-1}c^{-1}=1 \qquad \text{(lemma 1)}$$
$$\rightarrow \quad e^{-1}e^{-1}aba^{-1}\,\underline{\underline{c}}\,b^{-1}c^{-1}f^{-1}=1 \qquad \text{(cycling)}$$
$$\rightarrow \quad e^{-1}e^{-1}ab\,\underline{\underline{c}}\,a^{-1}b^{-1}c^{-1}f^{-1}=1 \qquad \text{(lemma 2)}$$
$$\rightarrow \quad e^{-1}e^{-1}a\,\underline{\underline{c}}\,ba^{-1}b^{-1}c^{-1}f^{-1}=1 \qquad \text{(lemma 2)}$$
$$\rightarrow \quad e^{-1}e^{-1}aba^{-1}b^{-1}cc^{-1}f^{-1}=1 \qquad \text{(lemma 2)}$$
$$\rightarrow \quad e^{-1}e^{-1}aba^{-1}b^{-1}f^{-1}=1$$
$$\rightarrow \quad e^{-1}e^{-1}aab^{-1}b^{-1}f^{-1}=1 \qquad \text{(Step 3: turning a handle into crosscaps)}$$
$$\rightarrow \quad e^{-1}e^{-1}aab^{-1}b^{-1}xf^{-1}x^{-1}=1 \qquad \text{(Step 4: introducing a cuff)}$$
$$\rightarrow \quad aa\,bb\,cc\,ede^{-1}=1 \qquad \text{(relabelling).}$$

The surface is $3\mathbb{R}P^2 \# D^2$.

12. (i) $T^2 \# pA \# qB = T^2 \# pT^2 \# pD^2 \# 2qT^2 \# 4qD^2 = (p+2q+1)T^2 \#(p+4q)D^2$

$3T^2 \# rA \# sB = 3T^2 \# rT^2 \# rD^2 \# 2sT^2 \# 4sD^2 = (r+2s+3)T^2 \#(r+4s)D^2$.

These two surfaces are homeomorphic

$\Leftrightarrow \quad p+2q+1=r+2s+3$ and $p+4q=r+4s$

$\Leftrightarrow \quad 2q-1=2s-3$ and $p+4q=r+4s$

$\Leftrightarrow \quad s=q+1$ and $r=p-4$.

Since p, q, r, s are non-negative integers, suitable values are:

$$p=4,\ q=0,\quad s=1,\ r=0.$$

[Check: $T^2 \# 4A = 5T^2 \# 4D^2 = 3T^2 \# B$.]

(ii) $(T^2 \# D^2) \# pA \# qB = T^2 \# D^2 \# pT^2 \# pD^2 \# 2qT^2 \# 4qD^2$

$$= (p+2q+1)T^2 \#(p+4q+1)D^2$$

$(3T^2 \# 2D^2) \# rA \# sB = 3T^2 \# 2D^2 \# rT^2 \# rD^2 \# 2sT^2 \# 4sD^2$

$$= (r+2s+3)T^2 \#(r+4s+2)D^2$$

The two surfaces are homeomorphic

$\Rightarrow \quad p+2q+1=r+2s+3$ and $p+4q+1=r+4s+2$

$\Rightarrow \quad 2q=2s-1$.

This last equation does not have integer solutions since, for $q, s \in \mathbb{Z}$, $2q$ is even, but $2s-1$ is odd.

Therefore $T^2 \# D^2 \mid 3T^2 \# 2D^2$.

13. **Conic (a)**

First we locate any finite branch points.

$$w^2 + (8z-2)w + (7z^2+z+1) = 0$$

$$\Rightarrow \quad w = \frac{-2(4z-1) \pm \sqrt{4(4z-1)^2 - 4(7z^2+z+1)}}{2} = (1-4z) \pm \sqrt{(4z-1)^2 - (7z^2+z+1)}.$$

This equation has equal roots if and only if

$$(4z-1)^2 - (7z^2+z+1) = 0$$

$$\Leftrightarrow \quad 9z^2 - 9z = 0$$

$$\Leftrightarrow \quad z = 0, 1.$$

Hence there are **two finite branch points**: (0, 1) and (1, –3).

Next, we determine which points lie over $z = \infty$.

Now (∞, w), $w \neq \infty$, lies on the locus if and only if $(\tilde{z}, w) = (0, w)$ lies on

$$\frac{7}{\tilde{z}^2} + \frac{8}{\tilde{z}}w + w^2 + \frac{1}{\tilde{z}} - 2w + 1 = 0$$

$$\text{i.e.} \quad 7 + 8\tilde{z}w + \tilde{z}^2 w^2 + \tilde{z} - 2\tilde{z}^2 w + \tilde{z}^2 = 0$$

which it does not.

Therefore there are no points (∞, w), $w \neq \infty$, on the locus. Hence there is only one point which lies over $z = \infty$, namely (∞, ∞). Since the locus has two finite branch points, (∞, ∞) must be a pinch point. [Recall that a conic has two branch points or none.]

Conic (a) belongs to category (iii): the branch points are (0, 1), (1, –3), the pinch point is (∞, ∞).

Conic (b)

To locate any finite branch points:

$$8w^2 + (8z-4)w + (2z^2+1) = 0$$

$$\Rightarrow \quad w = \frac{-4(2z-1) \pm \sqrt{16(2z-1)^2 - 32(2z^2+1)}}{32} = \frac{(1-2z) \pm \sqrt{(2z-1)^2 - 2(2z^2+1)}}{8}.$$

This equation has equal roots if and only if

$$(2z-1)^2 - 2(2z^2+1) = 0$$

$$\Rightarrow \quad -4z - 1 = 0$$

$$\Rightarrow \quad z = -\tfrac{1}{4}.$$

Hence there is only one finite branch point: $\left(-\frac{1}{4}, \frac{3}{8}\right)$.

Since a conic has either no branch points or two branch points, (∞, ∞) is also a branch point.

Therefore conic (b) belongs to category (ii): the branch points are $\left(-\frac{1}{4}, \frac{3}{8}\right)$, (∞, ∞).

Conic (c)

The equation is linear in w so there is a unique point on the locus for each z.

Therefore conic (c) belongs to category (i): there are no branch or pinch points, the conic is homeomorphic to the z-sphere.

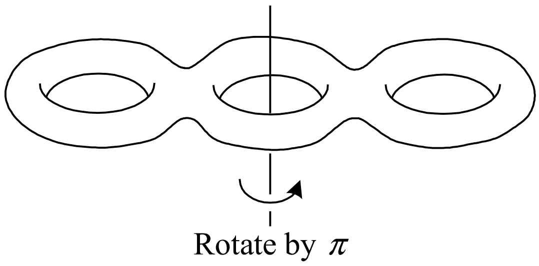
14. (i) $\chi(nT^2) = 2\chi(kT^2) - \delta \;\Rightarrow\; 2-2n = 2(2-2k) - \delta \;\Rightarrow\; n = 2k - 1 + \frac{1}{2}\delta$.

Since $\delta \geq 0$ we have $n \geq 2k - 1$.

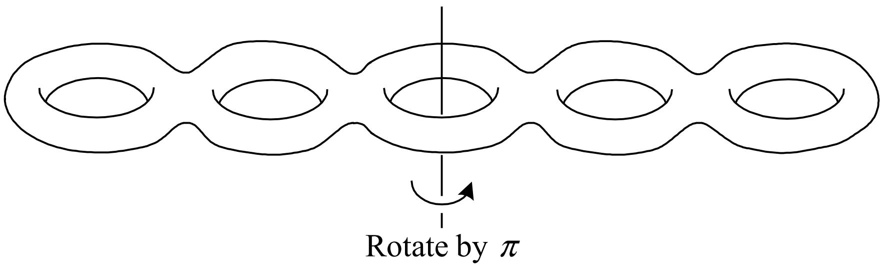
(ii) $\chi(nT^2) = r\chi(kT^2) - \delta \;\Rightarrow\; 2-2n = r(2-2k) - \delta \;\Rightarrow\; n = r(k-1) + 1 + \frac{1}{2}\delta$

Since $\delta \geq 0$ we have $n \geq r(k-1)+1$.

(iii) (a)

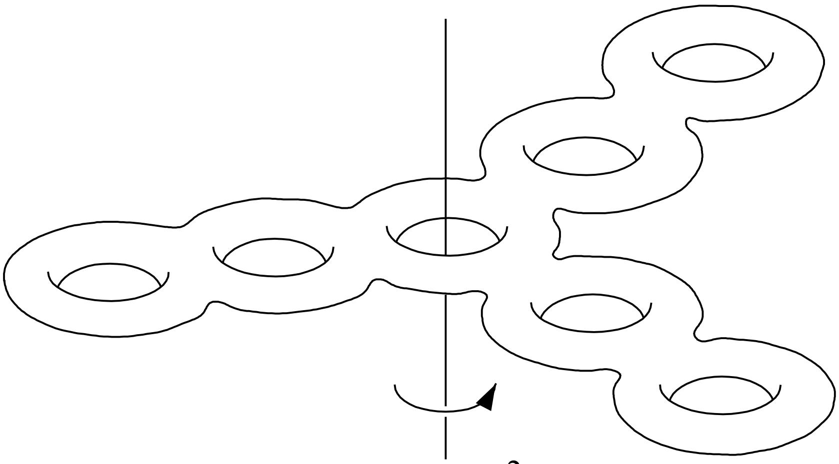


(b)

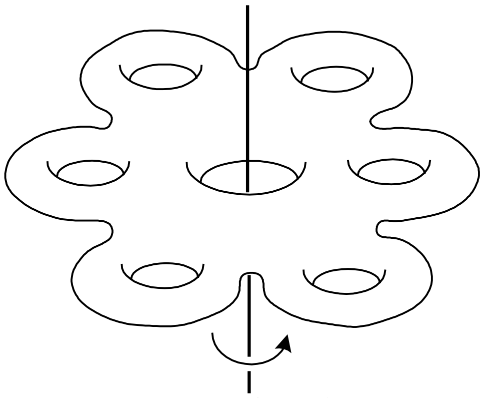


(iv) In part (iii) we have defined unbranched covers $3T^2 \xrightarrow{2} 2T^2$ and $5T^2 \xrightarrow{2} 3T^2$. Their composite $5T^2 \xrightarrow{2} 3T^2 \xrightarrow{2} 2T^2$ is an unbranched cover $5T^2 \xrightarrow{4} 2T^2$.

(v)



Rotate by $\frac{2\pi}{3}$



Rotate by $\pi/3$

15. Let $f(z) = z^6$, $g(z) = z + 4$, $h(z) = z^6 + 4$, $k(z) = 4$.

(i) On $|z| = 2$: $|g(z)| = |z+4| \le |z| + 4 = 6 < 64 = |z^6| = |f(z)|$.

Hence, by Rouché's theorem, $\omega(F) = \omega(f) = 6 \times 2\pi$.

Therefore, all 6 zeros of F lie in the disc $\{z \in \mathbb{C}: |z| \le 2\}$.

On $|z| = 1$: $|h(z)| = |z^6 + z| \le |z^6| + |z| = 2 < 4 = |k(z)|$.

Hence, by Rouché's theorem, $\omega(F) = \omega(k) = 0$.

Therefore, F has no zeros in the disc $\{z \in \mathbb{C}: |z| \le 1\}$ and so all 6 zeros of F lie in the annulus $\{z \in \mathbb{C}: 1 \le |z| \le 2\}$.

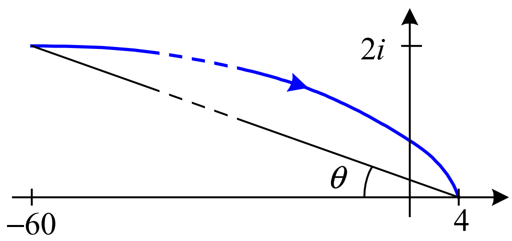
(ii) Let $x \in \mathbb{R}$. If $x \ge 0$ then $F(x) > 0$. If $-1 \le x < 0$ then x^6 and $x + 4$ are both positive so $F(x) > 0$. If $x < -1$ then $x^6 + x = x(x^5 + 1)$ and 4 are both positive so $F(x) > 0$.

Hence $F(x) > 0$ for all $x \in \mathbb{R}$ so F has no real zeros.

(iii) Let $y \in \mathbb{R}$. Then $F(iy) = (4 - y^6) + iy$. Since $4 - y^6$ and y cannot be zero simultaneously, $F(iy) \neq 0$, so F has no imaginary zeros.

(iv) $F(it) = (4 - t^6) + it$, $0 \leq t \leq 2$.

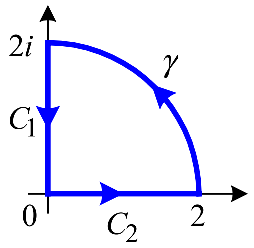
$F(0) = 4$, $F(2i) = -60 + 2i$; as t decreases $\operatorname{Re} F(it)$ increases and $\operatorname{Im} F(it) \geq 0$ and decreases.



$V_{C_1}(F) = -\pi + \theta$ where θ is small; hence

$V_{C_1}(F) \approx -\pi \quad \left(\text{and } -\pi < V_{C_1}(F) < -\dfrac{\pi}{2} \right)$.

(v)



On C_2: $F(z) \in \mathbb{R}$ for all $z \in C_2$ so $V_{C_2}(F) = 0$.

On γ: from part (i) and the generalisation of Rouché's theorem,

$$V_\gamma(F) = V_\gamma(f) + \varepsilon = 6 \times \frac{\pi}{2} + \varepsilon = 3\pi + \varepsilon \qquad \text{where } |\varepsilon| < \pi$$

$\Rightarrow \qquad 2\pi < V_\gamma(F) < 4\pi.$

Let $D = C_1 + C_2 + \gamma$. Then $\pi < V_D(F) < \dfrac{7\pi}{2}$; hence $V_D(F) = 2\pi$.

Therefore F has exactly one zero inside D.

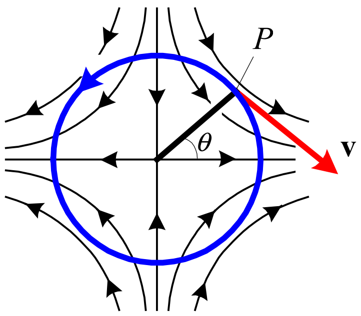
(vi) Since F has real coefficients and no real zeros, its zeros are complex conjugate pairs. From part (v), there is one zero in the first quadrant; therefore there is also a zero in the fourth quadrant.

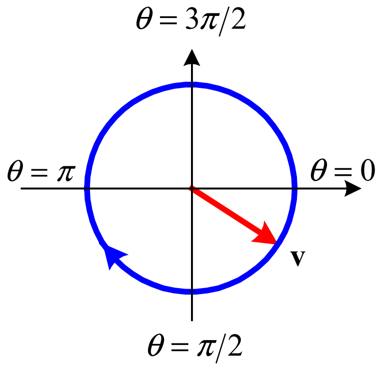
Since F has no purely imaginary zeros, the remaining zeros are two in each of the second and third quadrants.

In summary:

Quadrant:	I	II	III	IV
Number of zeros:	1	2	2	1

16. (i) (a) Consider a loop l winding once round the rest point in an anti-clockwise direction, as shown.

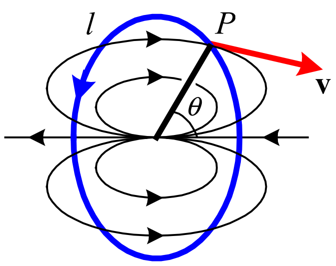


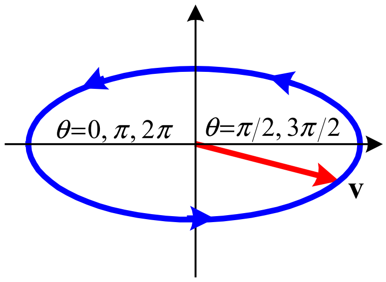


As the point P traces out the loop l, the 'sharp ends' of the translated vectors trace out a loop with winding number -1.

Therefore the rest point has index -1.

(b) Consider a loop l winding once round the rest point in an anti-clockwise direction, as shown.





As the point P traces out the loop l, the 'sharp ends' of the translated vectors trace out the indicated loop *twice* in the anti-clockwise direction; i.e. they trace out a loop with winding number 2.

Therefore the rest point has index 2.

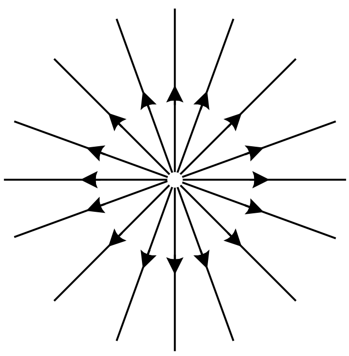
(ii) Index sum $= 2 \times (-1) + 2 = 0$.

Therefore, by the Poincaré index theorem, the closed orientable surface has $\chi = 0$ and is therefore a torus.

(iii) Index sum $= 4 \times (-1) + 2 = -2$.

Therefore, by the Poincaré index theorem, the closed orientable surface has $\chi = -2$ and is therefore a double torus, $2T^2$.

(iv)



(v) Index sum $= -1 + 2 + 1 = 2$.

By the Poincaré index theorem the surface has $\chi = 2$, so the surface is a sphere.