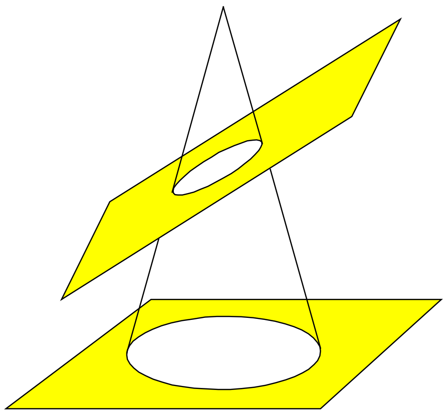
Review

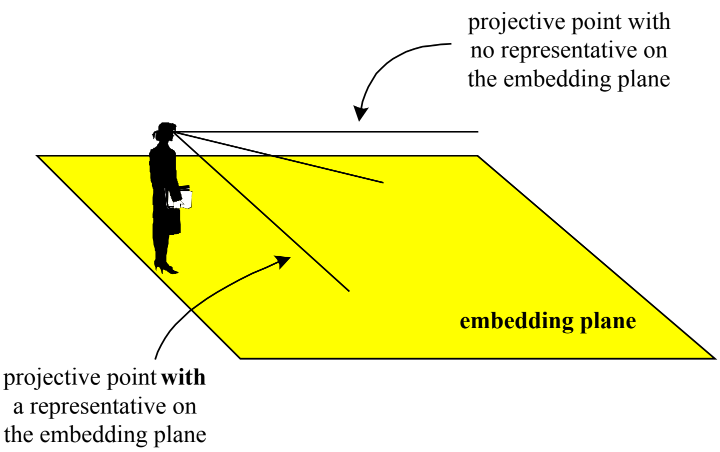
In projective geometry, we regard figures as being 'the same' if they can be made to *appear* the same as in the diagram below.



In projective geometry:

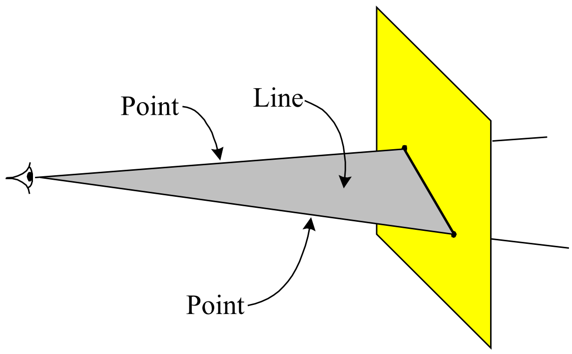
- a projective point – or Point – is a line through the origin
- a projective line – or Line – is a plane through the origin.

In order to obtain diagrams that we can draw and interpret in the usual way, we need to intersect these rays through the origin with a plane, called an **embedding plane**.

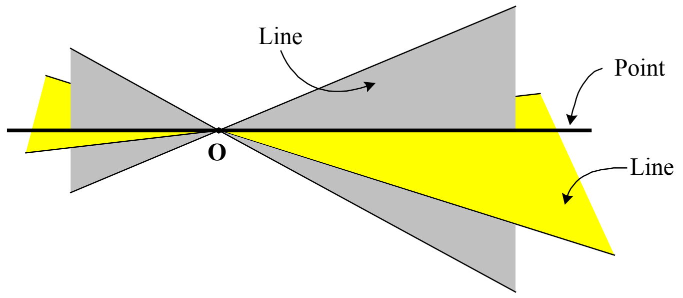


Properties of projective Points and Lines

Property 1 Any two distinct Points lie on (or determine) a unique Line.



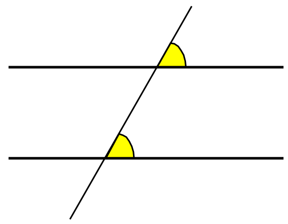
Property 2 Any two distinct Lines intersect in a unique Point.



Projective Geometry Theorems

In projective geometry, distances, angles and parallelism are *not* preserved, so any theorems which involve these notions are not theorems of projective geometry.

Example The following (add the words yourself!) is not a theorem of projective geometry.

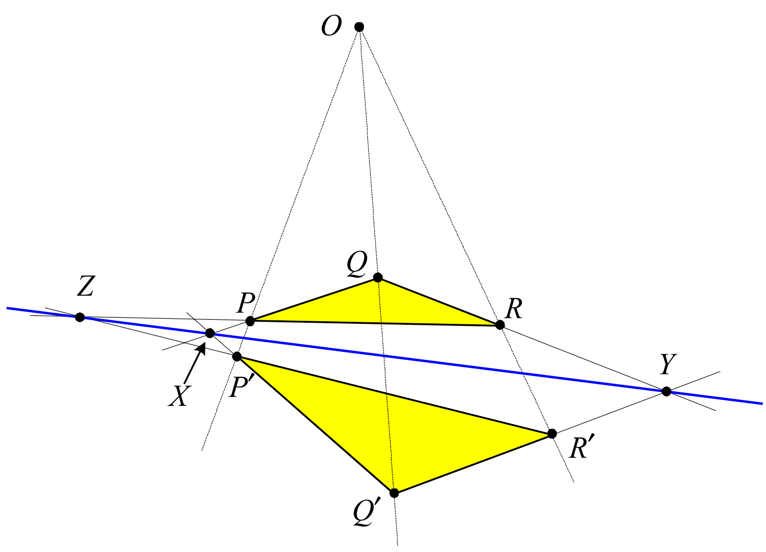


However, lines and intersections *are* preserved (provided we remember that any pair of lines intersect – even 'parallel' lines) so theorems that only involve these notions may be regarded as projective geometry theorems. We will look at two such theorems.

Desargues' Theorem

Let PQR and $P'Q'R'$ be two triangles such that the three lines PP', QQ' and RR' meet at a point O. Let PQ and $P'Q'$ meet at a point X
QR and $Q'R'$ meet at a point Y
PR and $P'R'$ meet at a point Z.

Then X, Y, Z are collinear.



Pappus' Theorem

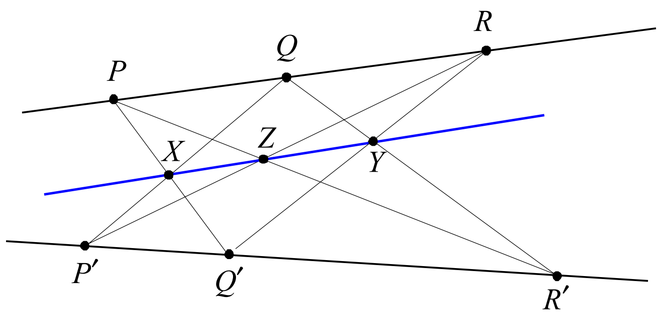
Let P, Q, R lie on a line and let P', Q', R' lie on a line. Let

PQ and $P'Q'$ meet at a point X

QR and $Q'R'$ meet at a point Y

PR and $P'R'$ meet at a point Z.

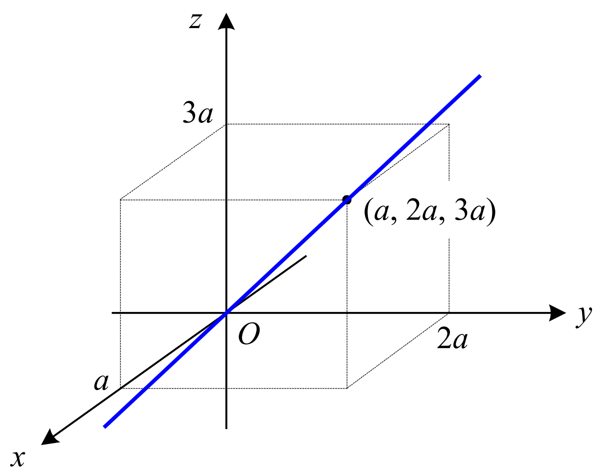
Then X, Y, Z are collinear.



Homogeneous Coordinates

In projective geometry, projective Points are lines through the origin.

Consider the line through the point $(1, 2, 3)$. Any (Euclidean) point on the line is of the form $(a, 2a, 3a)$ for some real number a. For example, $(1, 2, 3)$, $(3, 6, 9)$, $(-2, -4, -6)$, $(\pi, 2\pi, 3\pi)$ all lie on the line.

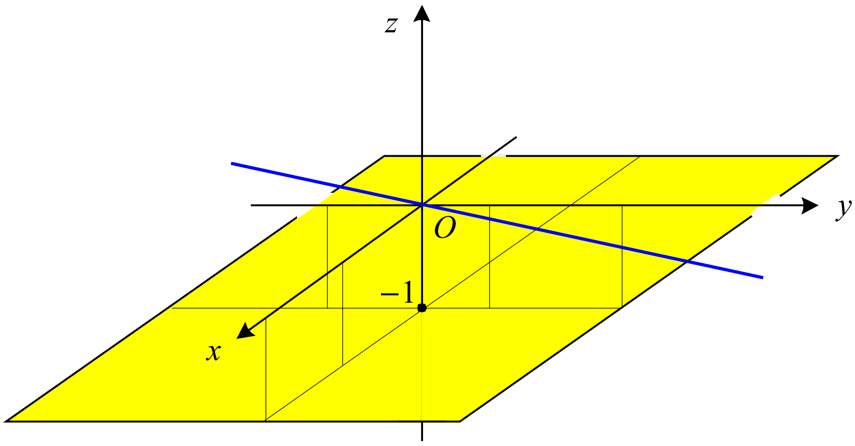


This projective Point is represented by $(a, 2a, 3a)$ for any real number a. We say the Point has **homogeneous coordinates** $[a, 2a, 3a]$ for any real number a. The homogeneous coordinates of a point are not uniquely determined – any non-zero multiple will do. Thus the Point above has homogeneous coordinates

$$[1, 2, 3] = [3, 6, 9] = [-2, -4, -6] = [\pi, 2\pi, 3\pi] = \ldots .$$

Question: where does this Point meet the embedding plane $z = 1$?

Question: if the embedding plane is the plane $z = -1$, what are the homogeneous coordinates of ideal points?



Question: which of the following homogeneous coordinates represent the same projective Point?

[1.5, 2, 2.5], [3, 5, 7], [3, 4, 5], [–6, –10, –14], [1, 5/3, 7/3], [2, 3, 4].

In summary:

a projective Point has homogeneous coordinates $[a, b, c]$

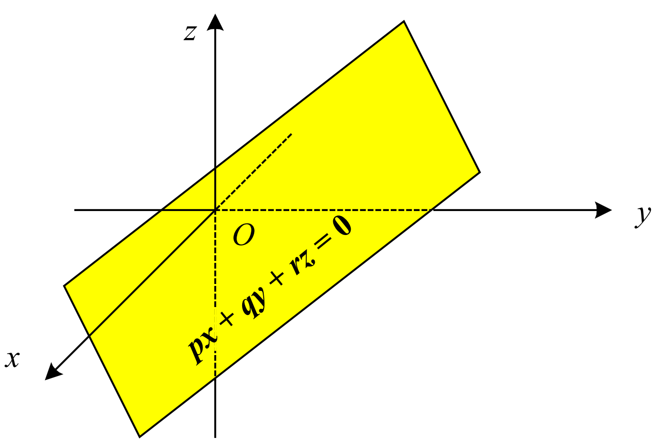
this represents the line through the origin and the point (a, b, c)

any non-zero multiple of the homogeneous coordinates are also homogeneous coordinates

$$[a, b, c] = [ka, kb, kc] \text{ for any } k \neq 0.$$

Projective lines

A plane through the origin has equation of the form $px + qy + rz = 0$.



If (x, y, z) satisfies $px + qy + rz = 0$ then so will (ka, kb, kc) for any $k \neq 0$. Therefore the equation of a projective Line is

$$px + qy + rz = 0$$

provided x, y, z represent *homogeneous coordinates* of a Point.

Examples

1. The xy-plane has equation $z = 0$.
   The xz-plane has equation $y = 0$.
   The yz-plane has equation $x = 0$.

2. The plane that passes through the origin and points $(1, 1, 0)$ and $(1, 0, 1)$ has equation

   $$x - y - z = 0.$$

   This is easy! *Any* equation of the form $px + qy + rz = 0$ which is satisfied by the coordinates $(1, 1, 0)$ and $(1, 0, 1)$ will do, so we just need to 'spot' a suitable equation.

3. Find the equation of the plane that passes through the origin and points $(1, 1, -2)$ and $(0, -1, 1)$.

Intersection Point of two projective Lines

To find the Point of intersection of two Lines, solve the equations of the Lines simultaneously.

Examples

1. Find the projective Point that is the intersection of the projective Lines

   $$x + y + z = 0 \quad \text{and} \quad 2x - y + 3z = 0.$$

   Solution

   $$\begin{array}{lr} & x + y + z = 0 \\ & 2x - y + 3z = 0 \\ \hline \text{Add} & 3x \quad\; + 4z = 0 \end{array}$$

   Let $z = 3$; then $x = -4$.
   Then, substituting into $x + y + z = 0$ gives $y = 1$.

   Therefore the Point of intersection is $[-4, 1, 3]$.

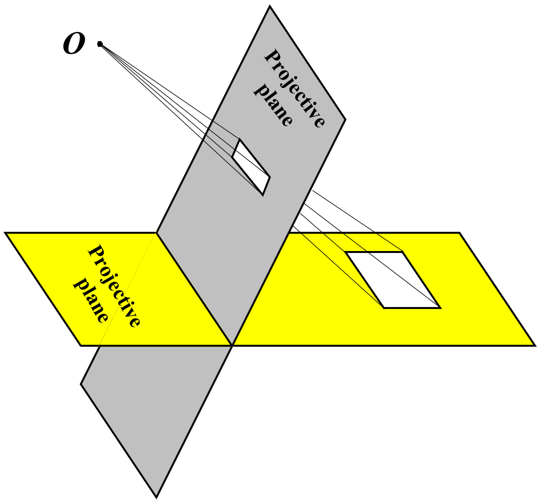
2. Find the projective Point that is the intersection of the projective Lines

   $$x - y + z = 0 \quad \text{and} \quad x + 3y - 2z = 0.$$

   Solution You should obtain $[1, -3, -4]$.

Projective transformations

A **perspectivity** from O is a mapping of the projective plane to itself such that P maps to P' when O, P, P' are collinear.



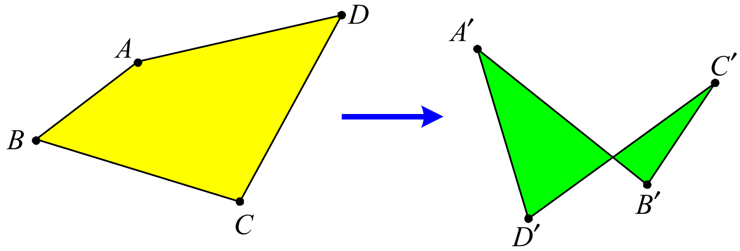
A **projective transformation** is a composition of perspectivities.

Compare with: a **Euclidean transformation** of the plane is a composition of translations, rotations and reflections.

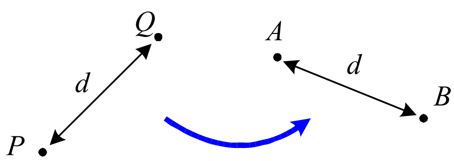
Fundamental theorem of Projective Geometry

Let $ABCD$ and $A'B'C'D'$ be any two quadrilaterals. Then there exists a unique projective transformation that maps $ABCD$ to $A'B'C'D'$. In other words, any four points (no three of which are collinear) map to any other four points (no three of which are collinear).

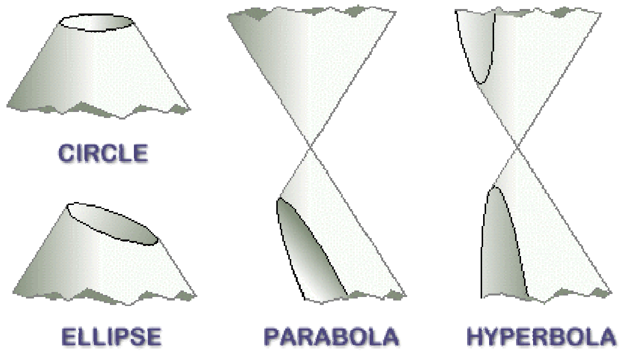
Colloquially: *all quadrilaterals are projectively equivalent*.



Compare with Euclidean geometry: transformations of Euclidean geometry can map any point to any other point (use a translation). But, for *pairs* of points, there exists a Euclidean transformation from one pair to another pair if and only if they are the same distance apart.

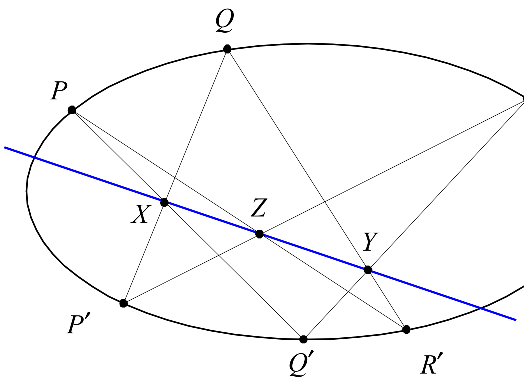


Conic sections



All conics are *projectively* the same – this is because each conic is the intersection of a (double) cone with some embedding plane. In other words, projective geometry cannot 'see' the difference between a line pair, a circle, an ellipse, a parabola and a hyperbola.

For Pappus' theorem this is great news! Since Pappus' theorem is a projective geometry theorem, once we have proved the theorem in the case of a line pair, we get *for free* the corresponding result for any conic. The following diagram illustrates the ellipse version of the theorem.



Proof of Pappus' Theorem.

Suppose we have the configuration show on page 3. Since $PQP'Q'$ is a quadrilateral, we can choose a projective transformation that maps it to any other quadrilateral. For simplicity, we shall map it to the quadrilateral with homogeneous coordinates

$$[1, 0, 0], \quad [0, 1, 0], \quad [0, 0, 1], \quad [1, 1, 1].$$

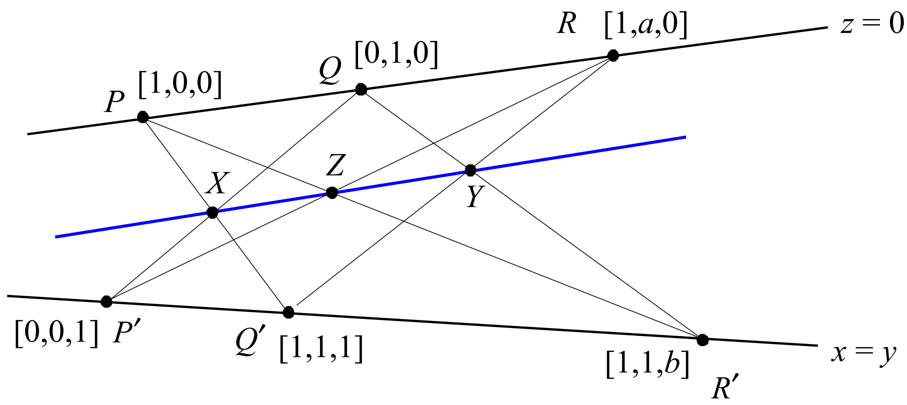
Thus we may assume that the figure is that given below. We shall calculate the homogeneous coordinates of Points and equations of Lines.

Line PQ P : $[1, 0, 0]$, Q : $[0, 1, 0]$. Therefore PQ has equation $z = 0$.

Point R R is a point on the line $z = 0$; hence z has coordinates $[\alpha, \beta, 0]$ or, if we scale suitably, $[1, a, 0]$

Line $P'Q'$ P' : $[0, 0, 1]$, Q' : $[1, 1, 1]$ so $P'Q'$ has equation $x - y = 0$ or $x = y$.

Point R' R' is a point on $x = y$. It has coordinates $[\alpha, \alpha, \beta] = [1, 1, b]$.



Line PQ' $P:[1,0,0]$, $Q':[1,1,1]$ so PQ' has equation

Line $P'Q$ $P':[0,0,1]$, $Q:[0,1,0]$ so $P'Q$ has equation

Point X PQ' has equation $P'Q$ has equation

X is the Point

Line PR' $P:[1,0,0]$, $R':[1,1,b]$ so PR' has equation

Line $P'R$ $P':[0,0,1]$, $R:[1,a,0]$ so $P'R$ has equation

Point Z PR' has equation $P'R$ has equation

Z is the Point

Line QR' $Q:[0,1,0]$, $R':[1,1,b]$ so QR' has equation

Line $Q'R$ $Q':[1,1,1]$, $R:[1,a,0]$ so $Q'R$ has equation

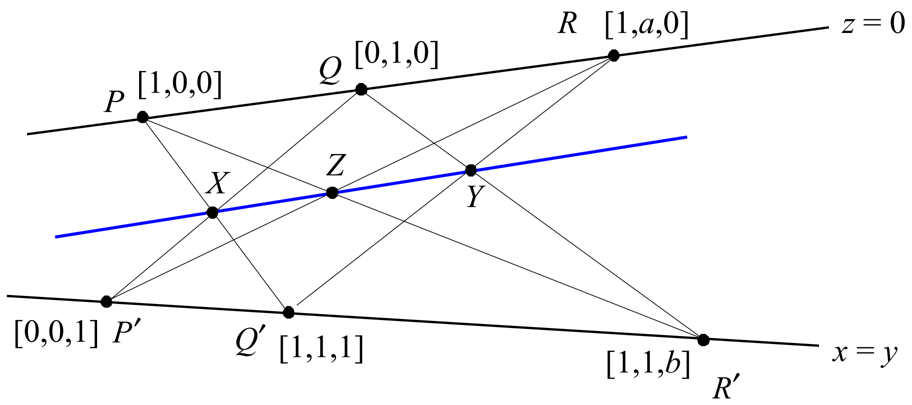
Point Y QR' has equation $Q'R$ has equation

Y is the Point

Line XZ $X:$ $Z:$ so XZ has equation $(ab-a)x+y-z=0$.

The crunch Does Y lie on the line XZ?

Exercises Construct Desargues' theorem in *Cabri*.
Construct Pappus' theorem (lines version) in *Cabri*.
Construct Pappus' theorem (general conic version) in *Cabri*.



Line PQ' $P : [1, 0, 0]$, $Q' : [1, 1, 1]$ so PQ' has equation $y = z$.

Line $P'Q$ $P' : [0, 0, 1]$, $Q : [0, 1, 0]$ so $P'Q$ has equation $x = 0$

Point X PQ' has equation $y = z$ $P'Q$ has equation $x = 0$

X is the Point $[0, 1, 1]$

Line PR' $P : [1, 0, 0]$, $R' : [1, 1, b]$ so PR' has equation $by = z$

Line $P'R$ $P' : [0, 0, 1]$, $R : [1, a, 0]$ so $P'R$ has equation $ax = y$

Point Z PR' has equation $by = z$ $P'R$ has equation $ax = y$

Z is the Point $[1, a, ab]$

Line QR' $Q : [0, 1, 0]$, $R' : [1, 1, b]$ so QR' has equation $bx = z$

Line $Q'R$ $Q' : [1, 1, 1]$, $R : [1, a, 0]$ so $Q'R$ has equation $y = ax + (1 - a)z$

Point Y QR' has equation $bx = z$ $Q'R$ has equation $y = ax + (1 - a)z$

$x = 1 \Rightarrow z = b \Rightarrow y = a + b - ab$

$\Rightarrow$ Y is the Point $[1, a + b - ab, b]$

Line XZ $X : [0, 1, 1]$, $Z : [1, a, ab]$ so XZ has equation $(ab - a)x + y - z = 0$.

The crunch Does Y lie on the line XZ?

$$(ab - a)x + y - z = ab - a + a + b - ab - b = 0$$

so Y lies on the line XZ.

Therefore X, Y, Z are collinear.